ARCHIVE

for

RATIONAL MECHANICS

and

ANALYSIS

Edited by

C. TRUESDELL

Volume 7, Number 1

SPRINGER-VERLAG
BERLIN - GÖTTINGEN - HEIDELBERG
(Postverlagsort Berlin · 24. 1. 1961)

Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité. EULER

... ut proinde his paucis consideratis tota haec materia redacta sit ad puram Geometriam, quod in physicis & mechanicis unice desideratum. LEIBNIZ

The ARCHIVE FOR RATIONAL MECHANICS AND ANALYSIS nourishes the discipline of mechanics as a deductive, mathematical science in the classical tradition and promotes pure analysis, particularly in contexts of application. Its purpose is to give rapid and full publication to researches of exceptional moment, depth, and permanence.

Each memoir must meet a standard of rigor set by the best work in its field. Contributions must consist largely in original research; on occasion, an expository paper may be invited.

English, French, German, Italian, and Latin are the languages of the Archive. Authors are urged to write clearly and well, avoiding an excessively condensed or crabbed style.

Manuscripts intended for the Archive should be submitted to an appropriate member of the Editorial Board.

The ARCHIVE FOR RATIONAL MECHANICS AND ANALYSIS appears in numbers struck off as the material reaches the press; five numbers constitute a volume. Subscriptions may be entered through any agent. The price is DM 96.— per volume.

For each paper 75 offprints are provided free of charge.

General Asymptotic Expansions of Laplace Integrals

A. Erdélyi

1. Introduction

It is well known [see*, for instance, Doetsch (1950—1956) vol. 2 Part I, Widder (1941) Chapter 5, Erdélyi (1956) sec. 2.2] that, under certain circumstances, an asymptotic expansion of $f(t)$ as $t \to 0+$ induces an asymptotic expansion of the Laplace transform

$$F(p) = \int_0^\infty e^{-pt} f(t)\, dt \tag{1.1}$$

as $p \to \infty$, and that a corresponding connection exists between asymptotic expansions of $f(t)$ as $t \to \infty$ on the one hand and of $F(p)$ as p approaches that singularity with the largest real part on the other hand.

The best known expansions in this connection are those involving powers of the variables, possibly multiplied by exponential functions. Other expansions have also been studied (see the references given above), but on the whole most, if not all, asymptotic expansions of Laplace transforms investigated are asymptotic expansions in the sense of Poincaré: the partial sums of the asymptotic expansions are linear combinations (with constant coefficients) of the functions of a given asymptotic sequence (scale, gauge).

Asymptotic expansions of a more general type have been known for some time [see, for instance, Schmidt (1936) or van der Corput (1955—1956)], and they have been used occasionally; but they have not been exploited for the asymptotic expansion of Laplace integrals. In recent years general asymptotic expansions found manifold applications, for instance, in certain singular perturbation problems [see, for instance, Kaplun & Lagerstrom (1957)] and in the study of coefficients of Laurent expansions [Wyman (1959) and unpublished work of the same author]. Their importance seems to be increasing, and it appears to be appropriate to apply them to the asymptotic expansion of Laplace integrals: this will be done in the present paper. The presentation follows that in an earlier note, Erdélyi (1947), on asymptotic expansions in the sense of Poincaré, and opportunity will be taken to remove some unnecessary restrictions imposed in that note and to correct an error.

The older results on asymptotic expansions of Laplace integrals have been applied successfully in more general methods of asymptotic expansions of functions defined by integrals, notably in Laplace's method and the method of

* References are listed at the end of this paper; they are quoted in the text by giving the name of the author followed by the year of publication in parentheses.

steepest descents [see, for instance, DOETSCH (1950—1956) vol. 2, chapter 3, ERDÉLYI (1956) chapter II]; and it is not unreasonable to assume that the results developed in this paper will permit of similar applications. (See also section 9 where an example of this occurs.)

2. Preliminaries

Let $f, g, \ldots, \varphi, \varphi_n, \ldots$ be functions of a variable z defined on a set R, and let z_0 be a limit point of R (z_0 itself may but need not belong to R). In our case z will be either t or p, so that z is a real or complex numerical variable, and it is clear what is meant by "limit point" and "neighborhood" in these cases. In the case of t, R will be the semi-infinite interval $t>0$ or $t\geqq 0$, and z_0 will be either 0 or ∞; in the case of p, R will usually be a region in the complex plane, and occasionally a semi-infinite interval on the real axis; in both of these cases z_0 will again be 0 or ∞.

We say that $f=O(\varphi)$ in R if there exists a constant (*i.e.*, a number independent of z) A so that $|f(z)|\leqq A|\varphi(z)|$ for all z in R; that $f=O(\varphi)$ as $z\to z_0$ if there exists a neighborhood U of z_0 and a constant A so that $|f(z)|\leqq A|\varphi(z)|$ for all z common to U and R; and that $f=o(\varphi(z))$ as $z\to z_0$ if corresponding to each $\varepsilon>0$ there exists a neighborhood U_ε of z_0 so that $|f(z)|\leqq \varepsilon|\varphi(z)|$ for all z common to U_ε and R.

Let $\{\varphi_n\}$ be a finite or infinite sequence of functions φ_n defined on R. We say that a relation involving the φ_n, and possibly other sequences of functions or numbers, holds *for all* n if it holds for all those n for which all terms occurring in that relation are defined. For instance, if the relation is $2\varphi_{n+1}=\varphi_n+\varphi_{n-1}$, and n runs through the sequence of non-negative integers, then "all n" means $n\geqq 1$; and if n runs through the integers $0, 1, \ldots, N$, then "all n" means, $1\leqq n\leqq N-1$.

We say that $\{\varphi_n\}$ is an *asymptotic sequence* if, for all n, $\varphi_{n+1}=o(\varphi_n)$ as $z\to z_0$. When it seems desirable, we say more specifically that $\{\varphi_n\}$ is an asymptotic sequence for $z\to z_0$; and occasionally we find it useful to indicate the range of n.

A sequence $\{\psi_n\}$ is said to *dominate* $\{\varphi_n\}$ if n ranges over the same set of values in the two sequences and if, for all n, $\varphi_n=O(\psi_n)$ as $z\to z_0$. These two sequences are said to be *equivalent* if each of them dominates the other. If one of two equivalent sequences is an asymptotic sequence, then the other will also be an asymptotic sequence.

The formal series $\sum f_n$ is said to be an *asymptotic expansion* of f with respect to the asymptotic sequence $\{\varphi_n\}$ if n ranges over the same values in the series $\sum f_n$ and the sequence $\{\varphi_n\}$ and if for each n,

$$f-\sum_{k\leqq n} f_k=o(\varphi_n) \quad \text{as} \quad z\to z_0.$$

(The sum occurring in this definition is a finite sum so that the left-hand side is a well defined function on R.) In this case we write

$$f\sim\sum f_n \qquad \{\varphi_n\} \quad \text{as} \quad z\to z_0. \tag{2.1}$$

Sometimes we shall write in greater detail "as $z\to z_0$ in R", and occasionally we shall omit the indication of the asymptotic sequence or the qualifying phrase "as $z\to z_0$".

The expansion (2.1) will be called an asymptotic expansion in the *restricted sense*, or in the sense of Poincaré, if for each n, $f_n = a_n \varphi_n$, where the a_n are constants.

The theory of asymptotic expansions in the restricted sense is well known. In particular, given f and $\{\varphi_n\}$, such an expansion, if it exists, is unique, and its coefficients can be determined recurrently by Poincaré's formula

$$a_n = \lim_{z \to z_0} \Big[f(z) - \sum_{k<n} a_k \varphi_k(z) \Big] \Big/ \varphi_n(z) .$$

The theory of the more general asymptotic expansions (2.1) resembles in many respects the theory of asymptotic expansions in Poincaré's sense. The most conspicuous difference between the two theories is a complete lack of uniqueness in the more general case. A function which possesses an asymptotic expansion with respect to an asymptotic sequence will clearly possess an infinity of asymptotic expansions, in the sense of (2.1), with respect to the same sequence; and any asymptotic expansion of a function will be an asymptotic expansion with respect to an infinity of asymptotic sequences, for instance with respect to all asymptotic sequences which dominate the one originally given.

Throughout this paper p is a complex variable, and we always write $p = \varrho + i\sigma$. If $f(t)$ is defined on $t > 0$, is integrable (in Lebesgue's sense) on the interval $0 < t < T$ for each $T > 0$, and if for a fixed value of p

$$F(p) = \lim_{T \to \infty} \int_0^T e^{-pt} f(t)\, dt \tag{2.2}$$

exists, we say that f possesses a Laplace transform for p and call $F(p)$ the *Laplace transform* of $f(t)$. We shall always denote functions and their Laplace transforms by corresponding lower case and capital letters.

We denote by L the class of all functions $f(t)$ which possess Laplace transforms for some p (this p may vary from function to function); by L_0 the class of the functions which possess Laplace transforms for each $p = \varrho > 0$; and by L_0' the class of those functions in L_0 which fail to possess a Laplace transform for $p = 0$.

If f possesses a Laplace transform for $p_0 = \varrho_0 + i\sigma_0$, then it will possess a Laplace transform for p whenever $\varrho > \varrho_0$, and $F(p)$ is an analytic function of p which is regular in the half-plane $\varrho > \varrho_0$. It will be useful to recall here some other elementary properties of Laplace transforms and to indicate the proofs of some of them.

Suppose that f is in L_0' and $f(t) \geqq 0$. Then $F(\varrho)$ exists, for $\varrho > 0$, not only as a limit, as in (2.2), but as a Lebesgue integral (1.1). Moreover, $F(\varrho)$ is clearly a decreasing function of ϱ. We wish to show that $F(\varrho)$ is unbounded, *i.e.*, $F(\varrho) \to +\infty$ as $\varrho \to 0+$ in this case. Indeed, if $F(\varrho) \leqq A$ for $\varrho > 0$, then also

$$F_T(\varrho) = \int_0^T e^{-\varrho t} f(t)\, dt \leqq A$$

for each $T > 0$ $\varrho > 0$. Since $F_T(\varrho)$ is a continuous function of ϱ for all real ϱ, $F_T(0) \leqq A$. Now $F_T(0)$ is an increasing function of T, and since this function is bounded (by A), $\lim F_T(0)$ exists as $T \to \infty$, so that f possesses a Laplace transform for $p = 0$ and is not in L_0'.

Next suppose that f is in L and $f(t) \geqq 0$. Then $F(\varrho)$ exists as a Lebesgue integral for sufficiently large values of ϱ and is a monotonic decreasing function of ϱ. $F(\varrho) \to 0$ as $\varrho \to +\infty$, but if $f(t) > 0$ in some neighborhood of $t=0$, then for each $\varepsilon > 0$, $e^{\varepsilon p} F(\varrho) \to \infty$ as $\varrho \to +\infty$. Indeed if $F(\varrho) \leqq A\, e^{-a\varrho}$ for some $A > 0$, $a > 0$ and all sufficiently large ϱ, then we can use Phragmén's inversion formula [Doetsch (1950—1956) vol. 1, p. 268]

$$\int_0^t f(u)\, du = \lim_{\varrho\to\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} f(n\,\varrho)\, e^{n\varrho t}$$

and for $0 < t < a$ obtain

$$\int_0^t f(u)\, du \leqq \lim_{\varrho\to\infty} \sum_{n=1}^{\infty} \frac{A}{n!} e^{-an\varrho + n\varrho t}$$

$$= \lim_{\varrho\to\infty} A\left[\exp e^{-(a-t)\varrho} - 1\right] = 0$$

so that $f(t) = 0$ almost everywhere in $0 < t < a$.

For an f in L, the Laplace transform need not converge absolutely for any value of p, and the integral in (1.1) may not exist as a Lebesgue integral. Nevertheless it is possible to transform (2.2), by integration by parts, into a Lebesgue integral. Indeed, if $F(\varrho)$ exists for $p = p_0 = \varrho_0 + i\sigma_0$ with $\varrho_0 > 0$, then

$$f_1(t) = \int_0^t f(u)\, du = o\left(e^{\varrho_0 t}\right) \quad \text{as} \quad t \to \infty, \tag{2.3}$$

and

$$F(p) = p \int_0^\infty e^{-pt} f_1(t)\, dt \qquad \varrho > \varrho_0. \tag{2.4}$$

where the integral on the right hand side is a Lebesgue integral.

3. The basic inequalities

The results on asymptotic expansions can be deduced from certain inequalities which are in essence comparison theorems showing that information on the relative behavior of two functions at $t=0$ or $t=\infty$ can be translated into information about the relative behavior of the Laplace transforms of these functions at $p=\infty$ and at the singularity with the largest real part respectively. The two functions will be called $g(t)$ and $h(t)$ and their respective Laplace transforms, $G(p)$ and $H(p)$.

The first of these inequalities correlates the behavior of an L-function as $t \to 0$ with the behavior of the Laplace transform as $p \to \infty$ in a certain region in the right half-plane.

Lemma 1. *Suppose that g and h are in L, $h(t) \geqq 0$, and $h(t) > 0$ for $0 < t < t_1$. Then*

$$\overline{\lim}\, \frac{|G(p)|}{H(\varrho)} \leqq \overline{\lim}\, \frac{|g(t)|}{h(t)} \tag{3.1}$$

when $t \to 0+$, and $p \to \infty$ in such a manner that $\varrho \to +\infty$ and for every fixed $\varepsilon > 0$, $p = o\left(\varrho\, H(\varrho)\, e^{\varepsilon\varrho}\right)$.

Before proving this lemma we note that the last condition is certainly satisfied if p is restricted to the sector

$$S_\Delta: \quad |\arg p| \leqq \tfrac{1}{2}\pi - \Delta \tag{3.2}$$

with $\Delta > 0$, for in this case $|p|/\varrho \leqq \operatorname{cosec} \Delta$, and $H(\varrho)\, e^{\varepsilon\varrho} \to \infty$ as $\varrho \to \infty$. The condition stated in the lemma allows, in some instances, $\arg p$ to approach $\pm\frac{1}{2}\pi$.

Proof. If g and h possess Laplace transforms for $\varrho_0 > 0$, then $G(p)$ and $H(p)$ exist, and $H(\varrho) > 0$, whenever $\varrho > \varrho_0$. With a fixed T, $0 < T < t_1$, we write $G(p) = I_1 + I_2$, where

$$I_1 = \int_0^T e^{-pt} g(t)\, dt,$$

$$I_2 = \lim_{A\to\infty} \int_T^A e^{-pt} g(t)\, dt.$$

As in (2.4), I_2 may be converted, by integration by parts, into an absolutely convergent integral,

$$I_2 = p \int_T^\infty e^{-pt} g_1(t)\, dt$$

where

$$g_1(t) = \int_T^t g(u)\, du = o(e^{\varrho_0 t}) \quad \text{as} \quad t \to \infty.$$

We set

$$U_T = \sup\left[\frac{|g(t)|}{h(t)} : 0 < t < T\right]$$

and assume that $U_T < \infty$ for some T. (Otherwise the right hand side of (3.1) is equal to $+\infty$, and there is nothing to prove.) Then $|g(t)| \leqq U_T\, h(t)$ for $0 < t < T$, and hence

$$|I_1| \leqq \int_0^T e^{-\varrho t}\, U_T\, h(t)\, dt \leqq U_T\, H(\varrho).$$

Moreover, $g_1(t)\, e^{-\varrho_0 t}$ is bounded when $t \geqq T$, say $|g_1(t)| \leqq B\, e^{\varrho_0 t}$. If we assume $\varrho > 2\varrho_0$ so that $\varrho - \varrho_0 > \frac{1}{2}\varrho$, we have

$$e^{-\varrho t}|g_1(t)| \leqq B\, e^{-\varrho t + \varrho_0 t} \leqq B\, e^{-\frac{1}{2}\varrho t}, \qquad t \geqq T, \quad \varrho > 2\varrho_0$$

so that

$$|I_2| \leqq \frac{2B|p|}{\varrho}\, e^{-\frac{1}{2}\varrho T}, \qquad \varrho > 2\varrho_0.$$

Thus for any $0 < T < t_1$ and $\varrho > 2\varrho_0$

$$\frac{|G(p)|}{H(\varrho)} \leqq U_T + \frac{2B|p|}{\varrho H(\varrho)}\, e^{-\frac{1}{2}\varrho T}.$$

Now let $p \to \infty$ in a manner satisfying the last condition of the lemma, and take $\varepsilon = \frac{1}{2}T$. Then the second term on the right hand side approaches zero and

$$\overline{\lim}\, \frac{|G(p)|}{H(\varrho)} \leqq U_T.$$

Since the left hand side here is independent of T, and the right hand side tends to the right hand side of (3.1) as $T \to 0+$, this proves the lemma.

The second inequality correlates the behavior of f as $t \to \infty$ with the behavior of F at a singularity with the largest real part. Since $F(p+c)$ is the Laplace transform of $e^{-ct} f(t)$, we may assume, without restricting the generality of the result, that the said singularity occurs at $p=0$, and this has been done in formulating the following result.

Lemma 2. *Suppose that g is in L_0 and h is in L_0', $h(t) \geqq 0$, and $h(t) > 0$ for $t > t_1 > 0$. Then the inequality* (3.1) *holds as $t \to +\infty$ and $p \to 0$ in the half-plane $\varrho > 0$.*

Proof. For any fixed $T > t_1$ set

$$U_T = \sup \left[\frac{|g(t)|}{h(t)} : t \geqq T \right],$$

and assume that $U_T < \infty$ for some T. (Otherwise there is nothing to prove.)

The integral defining $H(\varrho)$ for $\varrho > 0$ exists as a Lebesgue integral, and since $g(t)$ is dominated by $U_T h(t)$ when $t \geqq T$, the integral defining $G(p)$ for $\varrho > 0$ also exists as a Lebesgue integral. We again write $G(p) = I_1 + I_2$, where

$$I_1 = \int_0^T e^{-pt} g(t)\, dt, \qquad I_2 = \int_T^\infty e^{-pt} g(t)\, dt.$$

For $\varrho > 0$ we have

$$|I_1| \leqq \int_0^T |g(t)|\, dt = C_T,$$

say, and

$$|I_2| \leqq \int_T^\infty e^{-\varrho t} U_T h(t)\, dt \leqq U_T H(\varrho),$$

so that

$$\frac{|G(p)|}{H(\varrho)} \leqq \frac{C_T}{H(\varrho)} + U_T \qquad \varrho > 0.$$

Now, $H(\varrho) \to \infty$ as $\varrho \to 0+$, and hence

$$\overline{\lim} \frac{|G(p)|}{H(\varrho)} \leqq U_T$$

as $p \to 0$ in the half-plane $\varrho > 0$. Since the left hand side is independent of T, and the right hand side approaches the right hand side of (3.1) as $T \to \infty$, this proves lemma 2.

The condition $h(t) \geqq 0$ may be relaxed considerably since in the case of lemma 1 only small values of t, and in the case of lemma 2 only large values of t matter. Indeed, suppose that h_1 and h_2 are in L, and $h_1(t) = h_2(t) > 0$ for $0 < t < t_1$. Then

$$H_1(p) - H_2(p) = \int_{t_1}^\infty e^{-pt} [h_1(t) - h_2(t)]\, dt = O(e^{-\varrho t_1})$$

for all sufficiently large ϱ; and for $0 < \varepsilon < t_1$, we have

$$e^{\varepsilon\varrho} [H_1(\varrho) - H_2(\varrho)] \to 0 \quad \text{as} \quad \varrho \to +\infty,$$

while $e^{\varepsilon\varrho} H_1(\varrho)$ and $e^{\varepsilon\varrho} H_2(\varrho)$ are unbounded, so that

$$\frac{H_1(\varrho)}{H_2(\varrho)} = \frac{e^{\varepsilon\varrho} H_1(\varrho)}{\varepsilon^{\varepsilon\varrho} H_2(\varrho)} \to 1 \quad \text{as} \quad \varrho \to +\infty.$$

Again, suppose that h_1 and h_2 are in L_0', and $h_1(t)=h_2(t)>0$ for $t>t_1>0$. Then

$$H_1(p)-H_2(p)=\int_0^{t_1} e^{-pt}[h_1(t)-h_2(t)]\,dt$$

is an entire function of p which is bounded in some neighborhood of the origin, and since $H_1(\varrho)$ and $H_2(\varrho)$ are unbounded as $\varrho\to 0+$, we again have $H_1(\varrho)/H_2(\varrho)\to 1$ as $\varrho\to 0+$.

4. Asymptotic sequences

The results of the preceding section may be used to show that under certain circumstances the Laplace transforms $\{\Phi_n(p)\}$ of an asymptotic sequence $\{\varphi_n(t)\}$ form an asymptotic sequence.

Theorem 1. (i) *Suppose that* $\{\varphi_n(t)\}$ *is an asymptotic sequence for* $t\to 0+$ *such that for each* n, φ_n *is in* L, $\varphi_n(t)\geqq 0$, *and* $\varphi_n(t)>0$ *for* $0<t<t_n$. *Then* $\{\Phi_n(\varrho)\}$ *is an asymptotic sequence for* $p\to\infty$ *in such a manner that also* $\varrho\to+\infty$. (ii) *If, in addition, there is an unbounded set* R *in the* p*-plane such that* $\varrho\to+\infty$ *whenever* $p\to\infty$ *in* R, *and if for each* n, $\Phi_n(\varrho)=O(\Phi_n(p))$ *as* $p\to\infty$ *in* R, *then* $\{\Phi_n(p)\}$ *is also an asymptotic sequence as* $p\to\infty$ *in* R.

Proof. (i) If $p\to\infty$ so that $\varrho\to+\infty$, we have from lemma 1

$$\overline{\lim_{\varrho\to+\infty}}\ \frac{\Phi_{n+1}(\varrho)}{\Phi_n(\varrho)}\leqq\overline{\lim_{t\to 0+}}\ \frac{\varphi_{n+1}(t)}{\varphi_n(t)},$$

and the right hand side is equal to 0 since $\{\varphi_n(t)\}$ is an asymptotic sequence. Thus, $\Phi_{n+1}(\varrho)=o(\Phi_n(\varrho))$ as $p\to\infty$ in this case, and this proves the first half of the theorem.

(ii) Since $\varphi_n(t)\geqq 0$, we have in any event $|\Phi_n(p)|\leqq\Phi_n(\varrho)$, and hence $\Phi_n(p)=O(\Phi_n(\varrho))$, as $p\to\infty$ in R. Since by assumption also $\Phi_n(\varrho)=O(\Phi_n(p))$, it follows that the two sequences $\{\Phi_n(\varrho)\}$ and $\{\Phi_n(p)\}$ are equivalent as $p\to\infty$ in R. Since the first sequence is asymptotic, the second must also be asymptotic.

Theorem 2. (i) *Suppose that* $\{\varphi_n(t)\}$ *is an asymptotic sequence for* $t\to\infty$ *such that for each* n, φ_n *is in* L_0', $\varphi_n(t)\geqq 0$, *and* $\varphi_n(t)>0$ *for* $t>t_n$. *Then* $\{\Phi_n(\varrho)\}$ *is an asymptotic sequence for* $p\to 0$ *in the half-plane* $\varrho>0$. (ii) *If, in addition, there is a set* R *in the half-plane* $\varrho>0$ *such that* $p=0$ *is a limit point of* R *and if for each* n, $\Phi_n(\varrho)=O(\Phi_n(p))$ *as* $p\to 0$ *in* R, *then* $\{\Phi_n(p)\}$ *is also an asymptotic sequence for* $p\to 0$ *in* R.

Proof. From Lemma 2,

$$\overline{\lim_{\varrho\to 0+}}\ \frac{\Phi_{n+1}(\varrho)}{\Phi_n(\varrho)}\leqq\overline{\lim_{t\to+\infty}}\ \frac{\varphi_{n+1}(t)}{\varphi_n(t)},$$

and the proof is quite similar to that of theorem 1.

5. Asymptotic expansions

The results of section 3 may also be used to deduce asymptotic expansions of Laplace transforms. The technique is well known, and one arrives in this manner at results which establish under very general circumstances what Doetsch [(1950—1956) vol. 2, p. 43] calls the "ideal case of Abelian asymptotics".

Theorem 3. *Let $\{\psi_n(t)\}$ be an asymptotic sequence, for $t \to 0+$, which is equivalent to a sequence $\{\varphi_n(t)\}$ satisfying the conditions of theorem* 1 (i); *let R be an unbounded set in the p-plane such that $\varrho \to +\infty$, and for each n and each $\varepsilon > 0$*

$$\frac{p}{\varrho\, \Phi_n(\varrho)\, e^{\varepsilon\varrho}} \to 0 \tag{5.1}$$

as $p \to \infty$ in R; and let $\{X_n(p)\}$ be a sequence equivalent to $\{\Phi_n(\varrho)\}$ as $p \to \infty$ in R. If, under these circumstances,

$$f(t) \sim \sum f_n(t) \qquad \{\psi_n(t)\} \tag{5.2}$$

as $t \to 0+$, where f and all the f_n are in L, then

$$F(p) \sim \sum F_n(p) \qquad \{X_n(p)\} \tag{5.3}$$

as $p \to \infty$ in R.

Proof. Fix n and set

$$g = f - \sum_{k \leq n} f_k, \qquad h = \varphi_n. \tag{5.4}$$

Then g and h are clearly in L, $\varphi_n(t) \geqq 0$, and $\varphi_n(t) > 0$ for $0 < t < t_n$. Thus, the conditions of lemma 1 are satisfied and, moreover, for each $\varepsilon > 0$, $p = o\left(\varrho\, \Phi_n(\varrho)\, e^{\varepsilon\varrho}\right)$ by (5.1) as $p \to \infty$ in R. It is a consequence of (5.2) that $g(t) = o\left(\psi_n(t)\right)$ as $t \to 0+$, and since $\{\psi_n(t)\}$ and $\{\varphi_n(t)\}$ are equivalent, also $g(t) = o\left(\varphi_n(t)\right)$ as $t \to 0+$. It follows that the right hand side of (3.1) vanishes, and consequently also $\lim G(p)/\Phi_n(\varrho) = 0$ as $p \to \infty$ in R. But then $G(p) = o\left(\Phi_n(\varrho)\right)$, and since $\{\Phi_n(\varrho)\}$ and $\{X_n(p)\}$ are equivalent also $G(p) = o\left(X_n(p)\right)$, as $p \to \infty$ in R. Since this holds for each n, we have (5.3). Note that $\{\Phi_n(\varrho)\}$ is an asymptotic sequence by theorem 1, and hence also $\{X_n(p)\}$ is an asymptotic sequence*.

At first the infinite set of conditions (5.1) might seem to restrict R rather severely but this is not really the case. As has been pointed out in section 3, a sector S_Δ, $\Delta > 0$, is always an admissible R, and in special cases larger regions R might be admissible.

Theorem 4. *Let $\{\psi_n(t)\}$ be an asymptotic sequence, for $t \to \infty$, which is equivalent to a sequence $\{\varphi_n(t)\}$ satisfying the conditions of theorem* 2 (i); *let R be a set in the half-plane $\varrho > 0$ such that $p = 0$ is a limit point of R; and let $\{X_n(p)\}$ be a sequence equivalent to $\{\Phi_n(\varrho)\}$ as $p \to 0$ in R. If, under these circumstances,* (5.2) *holds as $t \to \infty$, and if f and all the f_n are in L_0, then* (5.3) *holds as $p \to 0$ in R.*

Proof. Fix n and define g and h by (5.4). Clearly g is in L_0 and h is in L_0', $\varphi_n(t) \geqq 0$, and $\varphi_n(t) > 0$ for $t > t_n$. Thus, the conditions of lemma 2 are satisfied; and from this point the proof resembles closely the proof of theorem 3.

6. Sequences of powers

In the remaining sections we shall give some examples and applications of the general results, and will comment on some points arising from the comparison of our examples with standard results.

---

* The corresponding results in ERDÉLYI (1947), theorems 1 and 2, are defective in that the sequence denoted there by $\{\psi_n(p)\}$ need not be an asymptotic sequence. The corrected form appears in ERDÉLYI (1956) p. 31f.

The first, and most important, example concerns powers of the variables. With complex parameters λ_n we set

$$\psi_n(t) = t^{\lambda_n - 1}, \quad \varphi_n(t) = t^{\operatorname{Re}\lambda_n - 1}, \qquad \operatorname{Re}\lambda_n > 0.$$

Clearly, $\{\psi_n(t)\}$ and $\{\varphi_n(t)\}$ are equivalent sequences on the semi-infinite interval $t > 0$. The φ_n certainly possess Laplace transforms

$$\Phi_n(p) = \int_0^\infty e^{-pt} t^{\operatorname{Re}\lambda_n - 1}\, dt = \Gamma(\operatorname{Re}\lambda_n)\, p^{-\operatorname{Re}\lambda_n}$$

for all $\varrho > 0$, and fail to possess Laplace transforms for $p = 0$, so that all φ_n are in L_0'. Moreover, $\varphi_n(t) > 0$ for $t > 0$ and all n so that both lemma 1 and lemma 2 apply here.

If the $\operatorname{Re}\lambda_n$ form an increasing (decreasing) sequence, both $\{\psi_n(t)\}$ and $\{\varphi_n(t)\}$ are asymptotic sequences for $t \to 0+$ $(t \to \infty)$, and we can apply theorem 3 (theorem 4). In view of the special form of $\Phi_n(\varrho)$, (5.1) is satisfied if and only if $p e^{-\varepsilon\varrho} \to 0$ for each fixed $\varepsilon > 0$.

From now on R_0 will always stand for any unbounded set in the p-plane such that $\varrho \to +\infty$ and for every $\varepsilon > 0$ $p e^{-\varepsilon\varrho} \to 0$, as $p \to \infty$ in R_0. We shall continue to use the notation S_Δ for the sector defined in (3.2).

We also set

$$X_n(p) = p^{-\lambda_n}.$$

Now, writing $p = |p|\, e^{i\vartheta}$, $\varrho = |p| \cos\vartheta$, we have

$$\frac{\Phi_n(\varrho)}{|X_n(p)|} = \Gamma(\operatorname{Re}\lambda_n) \frac{|p^{\lambda_n}|}{\varrho^{\operatorname{Re}\lambda_n}} = \Gamma(\operatorname{Re}\lambda_n) \frac{e^{\vartheta \operatorname{Im}\lambda_n}}{(\cos\vartheta)^{\operatorname{Re}\lambda_n}},$$

and see that this quotient is bounded if and only if $\sec\vartheta$ is bounded, *i.e.*, if p is restricted to S_Δ with some $\Delta > 0$. Since the above quotient is always bounded away from zero, it follows that $\{\Psi_n(\varrho)\}$ and $\{X_n(p)\}$ are equivalent in every S_Δ with $\Delta > 0$, and in no larger region, as $p \to 0$ or $p \to \infty$.

From theorems 3 and 4 we then obtain the following results:

Theorem 5. *For each n, let* $\operatorname{Re}\lambda_n > 0$, $\operatorname{Re}\lambda_n > \operatorname{Re}\lambda_{n-1}$. *If*

$$f(t) \sim \sum f_n(t) \qquad \{t^{\lambda_n - 1}\} \tag{6.1}$$

as $t \to 0+$, and if f and all the f_n are in L; then

$$F(p) \sim \sum F_n(p) \qquad \{\varrho^{-\operatorname{Re}\lambda_n}\} \tag{6.2}$$

as $p \to \infty$ in some R_0, and

$$F(p) \sim \sum F_n(p) \qquad \{p^{-\lambda_n}\} \tag{6.3}$$

as $p \to \infty$ in some S_Δ, $\Delta > 0$.

Theorem 6. *For each n let* $\operatorname{Re}\lambda_n > 0$, $\operatorname{Re}\lambda_n < \operatorname{Re}\lambda_{n-1}$. *If* (6.1) *holds as $t \to +\infty$, and if f and all the f_n are in L_0; then* (6.2) *holds as $p \to 0$ in the half-plane $\varrho > 0$, and* (6.3) *holds as $p \to 0$ in some S_Δ, $\Delta > 0$.*

Note that the asymptotic expansion $F \sim \sum F_n$ has been established here in two different senses: (6.2) has a larger region of validity, but (6.3) has a better asymptotic sequence. This flexibility is lost if one restricts oneself to asymptotic

expansions in the sense of Poincaré, for (6.2) can never be such an expansion [see (6.5) below].

The best known case of these expansions is the particular case in which (6.1) assumes the form

$$f(t) \sim \sum c_n t^{\lambda_n - 1} \qquad \{t^{\lambda_n - 1}\} \tag{6.4}$$

where the c_n are independent of t, and (6.2) and (6.3) become

$$F(p) \sim \sum c_n \Gamma(\lambda_n)\, p^{-\lambda_n} \qquad \{\varrho^{-\operatorname{Re}\lambda_n}\} \quad \text{or} \quad \{p^{-\lambda_n}\} \tag{6.5}$$

as the case may be.

Other special cases of theorem 5 involve inverse factorial series of various kinds. With a non-negative integer n, we have

$$\int_0^\infty e^{-pt}(1-e^{-t})^n\, dt = \frac{n!}{p(p+1)\ldots(p+n)} \qquad \varrho > 0,$$

$$\int_0^\infty e^{-pt}(e^t-1)^n\, dt = \frac{n!}{p(p-1)\ldots(p-n)} \qquad \varrho > n,$$

$$\int_0^\infty e^{-pt}(e^{\frac{1}{2}t}-e^{-\frac{1}{2}t})^{2n}\, dt = \frac{(2n)!}{(p-n)(p-n+1)\ldots(p+n)} \qquad \varrho > n.$$

Moreover, the three sequences $\{\psi_n(t)\}$ involved here are equivalent, respectively, to $\{t^n\}$, $\{t^n\}$, $\{t^{2n}\}$ as $t \to 0+$, and their Laplace transforms are correspondingly equivalent to $\{p^{-n-1}\}$, $\{p^{-n-1}\}$, $\{p^{-2n-1}\}$ as $p \to \infty$. With constant coefficients, c_n, we thus obtain the following examples to theorem 5:

If $f(t)$ is in L, and if

$$f(t) \sim \sum c_n (1-e^{-t})^n \qquad \{t^n\} \quad \text{as} \quad t \to 0+, \tag{6.6}$$

then

$$F(p) \sim \sum \frac{n!\, c_n}{p(p+1)\ldots(p+n)} \qquad \{\varrho^{-n-1}\} \quad \text{as} \quad p \to \infty \text{ in } R_0 \tag{6.7}$$

(inverse factorial series); if

$$f(t) \sim \sum c_n (e^t-1)^n \qquad \{t^n\} \quad \text{as} \quad t \to 0+, \tag{6.8}$$

then

$$F(p) \sim \sum \frac{n!\, c_n}{p(p-1)\ldots(p-n)} \qquad \{\varrho^{-n-1}\} \quad \text{as} \quad p \to \infty \text{ in } R_0 \tag{6.9}$$

(inverse backward factorial series); and if

$$f(t) \sim \sum c_n (2\sin\tfrac{1}{2}t)^n \qquad \{t^{2n}\} \quad \text{as} \quad t \to 0+, \tag{6.10}$$

then

$$F(p) \sim \sum \frac{(2n)!\, c_n}{(p-n)(p-n+1)\ldots(p+n)} \qquad \{\varrho^{-2n-1}\} \quad \text{as} \quad p \to \infty \text{ in } R_0 \tag{6.11}$$

(inverse central factorial series).

(6.4) is an asymptotic expansion in the sense of Poincaré, and so is (6.5) if the second asymptotic sequence is used and accordingly p is restricted to S_Δ,

$\Delta > 0$. The expansions (6.6) to (6.11) may be rewritten as asymptotic expansions in the sense of Poincaré by replacing the asymptotic sequences indicated in $\{\ \}$ by the equivalent sequences formed by the functions occurring in the expansions themselves, and again restricting p to S_Δ, $\Delta > 0$. Thus, these expansions, at least in S_Δ, enjoy uniqueness properties not available in the general case.

7. Some special Laplace integrals involving logarithms

For the discussion of asymptotic expansions involving logarithms, we require certain results on the Laplace transforms of functions like $t^{\lambda-1}(\pm\log t)^\alpha$. Since these results are useful in several connections, they will be given in more detail than is required for the application in section 8.

First we discuss the function

$$L(\varrho, \lambda, \alpha) = \int_0^c (-\log t)^\alpha t^{\lambda-1} e^{-\varrho t}\, dt, \tag{7.1}$$

assuming that $0 < c < 1$, $\lambda > 0$, α real; and investigate the behavior of this function as $\varrho \to +\infty$.

Differentiating

$$\int_0^\infty t^{\lambda-1} e^{-\varrho t}\, dt = \Gamma(\lambda)\, \varrho^{-\lambda} \qquad \lambda > 0, \quad \varrho > 0$$

k times with respect to $\pm\lambda$, we first obtain

$$\int_0^\infty (\pm\log t)^k t^{\lambda-1} e^{-\varrho t}\, dt = \sum_{r=0}^k (\pm 1)^r \binom{k}{r} \Gamma^{(r)}(\lambda)\, \varrho^{-\lambda} (\mp\log\varrho)^{k-r} \quad \lambda > 0,\ \varrho > 0 \tag{7.2}$$

where either all upper or all lower signs must be taken.

Next, we wish to show that

$$\int_c^\infty (-\log t)^k t^{\lambda-1} e^{-\varrho t}\, dt = O\left((2\lambda)^\lambda \varrho^{-\lambda-1} e^{-\frac{1}{2}c\varrho}\right), \tag{7.3}$$

uniformly in λ for $\lambda > 0$, as $\varrho \to +\infty$. To show this, we remark that $|\log t|^k \leqq A\,t$ for some $A > 0$ and all $t \geqq c$, so that the integral on the left hand side of (7.3) is dominated by

$$A \int_c^\infty t^\lambda e^{-\varrho t}\, dt.$$

Furthermore, $\varrho t/(2\lambda) < \exp \varrho t/(2\lambda)$ so that the last integral in its turn is dominated by

$$A\left(\frac{2\lambda}{\varrho}\right)^\lambda \int_c^\infty e^{-\frac{1}{2}\varrho t}\, dt = \frac{2A(2\lambda)^\lambda}{\varrho^{\lambda+1}} e^{-\frac{1}{2}c\varrho}.$$

This proves (7.3) since A is clearly independent of λ.

From (7.2) and (7.3) we have for any non-negative integer k,

$$L(\varrho, \lambda, k) = \sum_{r=0}^k (-1)^r \binom{k}{r} \Gamma^{(r)}(\lambda)\, \varrho^{-\lambda} (\log\varrho)^{k-r} + O\left((2\lambda)^\lambda \varrho^{-\lambda-1} e^{-\frac{1}{2}c\varrho}\right), \tag{7.4}$$

uniformly in λ, as $\varrho \to +\infty$.

We wish to extend this result from integer to arbitrary real values of α. This extension will be based on the functional equation

$$\frac{1}{\Gamma(\beta)} \int_0^\infty z^{\beta-1} L(\varrho, \lambda + z, \gamma)\, dz = L(\varrho, \lambda, \gamma - \beta), \qquad \beta > 0, \quad \lambda > 0 \tag{7.5}$$

which is easily proved by substituting the basic integral (7.1) for $L(\varrho, \lambda + z, \gamma)$ on the left hand side of (7.5) and interchanging the orders of integration. Since $c < 1$, the double integral is absolutely convergent and can be evaluated as a repeated integral.

Lemma 3. *For $\lambda > 0$, α real, and $\varrho \to +\infty$,*

$$L(\varrho, \lambda, \alpha) \sim \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} \Gamma^{(n)}(\lambda)\, \varrho^{-\lambda} (\log \varrho)^{\alpha - n} \qquad \{\varrho^{-\lambda} (\log \varrho)^{\alpha - n}\}.$$

Proof. Choose a positive integer $k > \alpha$: we shall show that the asymptotic expansion stated in lemma 3 holds to $k+1$ terms. Since k can be arbitrarily large, this proves lemma 3.

We have $\alpha = k - \beta$ where k is a positive integer, and $\beta > 0$. From (7.5),

$$L(\varrho, \lambda, \alpha) = \frac{1}{\Gamma(\beta)} \left(\int_0^Z + \int_Z^\infty \right) z^{\beta-1} L(\varrho, \lambda + z, k)\, dz = L_1 + L_2, \tag{7.6}$$

say, where Z is an arbitrary positive number.

For $t \leqq c < 1$,

$$t^{-Z} \int_Z^\infty z^{\beta-1} t^z\, dz = \int_Z^\infty z^{\beta-1} e^{(z-Z)\log t}\, dz \leqq \int_Z^\infty z^{\beta-1} e^{-(z-Z)|\log c|}\, dz$$

is a bounded function of t so that

$$L_2 = O\left(\int_0^c (-\log t)^k\, t^{\lambda + Z - 1} e^{-\varrho t}\, dt \right) \qquad \varrho \to \infty$$

and by (7.4),

$$L_2 = O\left(\varrho^{-\lambda - Z} (\log \varrho)^k\right) \qquad \varrho \to \infty. \tag{7.7}$$

We also have

$$\frac{1}{\Gamma(\beta)} \int_0^Z z^{\beta-1} O\left((2\lambda + 2z)^{\lambda+z} \varrho^{-\lambda-z-1} e^{-\frac{1}{2}c\varrho}\right) dz = O\left(\varrho^{-\lambda-1} e^{-\frac{1}{2}c\varrho}\right) \qquad \varrho \to \infty, \tag{7.8}$$

and hence, substituting (7.4) in L_1,

$$\begin{aligned} L(\varrho, \lambda, \alpha) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} (\log \varrho)^{k-r} \frac{1}{\Gamma(\beta)} \int_0^Z z^{\beta-1} \Gamma^{(r)}(\lambda + z)\, \varrho^{-\lambda-z}\, dz\, + \\ + O\left(\varrho^{-\lambda-Z} (\log \varrho)^k\right), \end{aligned} \tag{7.9}$$

since the O-term appearing in (7.8) may be included in that appearing in (7.7).

Further progress now depends on the asymptotic expansion of the integrals

$$\frac{1}{\Gamma(\beta)} \int_0^Z z^{\beta-1} \Gamma^{(r)}(\lambda + z)\, \varrho^{-\lambda-z}\, dz.$$

Apart from the factor $\varrho^{-\lambda}$, these are Laplace integrals with the parameter $\log \varrho$, and their asymptotic expansion can be deduced from theorem 5. From

$$z^{\beta-1}\Gamma^{(r)}(\lambda+z)=\sum_{s=0}^{k-r}\frac{1}{s!}\Gamma^{(r+s)}(\lambda)\,z^{\beta+s-1}+o\left(z^{\beta+k-r-1}\right)$$

as $z\to 0$ it follows by theorem 5 that

$$\frac{1}{\Gamma(\beta)}\int_0^Z z^{\beta-1}\Gamma^{(r)}(\lambda+z)\,\varrho^{-\lambda}e^{-z\log\varrho}\,dz$$
$$=\sum_{s=0}^{k-r}\frac{\Gamma(\beta+s)}{s!\,\Gamma(\beta)}\Gamma^{(r+s)}(\lambda)\,\varrho^{-\lambda}(\log\varrho)^{-\beta-s}+o\left(\varrho^{-\lambda}(\log\varrho)^{-\beta-k+r}\right)$$

as $\varrho\to+\infty$. We now substitute this expansion in (7.9), reduce the double sum to a single sum by means of Vandermonde's identity

$$\sum_{r+s=n}\binom{k}{r}\binom{-\beta}{s}=\binom{k-\beta}{n}=\binom{\alpha}{n},$$

and obtain

$$L(\varrho,\lambda,\alpha)=\sum_{n=0}^{k}(-1)^n\binom{\alpha}{n}\Gamma^{(n)}(\lambda)\,\varrho^{-\lambda}(\log\varrho)^{\alpha-n}+o\left(\varrho^{-\lambda}(\log\varrho)^{\alpha-k}\right)\qquad(7.10)$$

as $\varrho\to+\infty$, thus establishing lemma 3.

Secondly we discuss the function

$$M(\varrho,\lambda,\alpha)=\int_c^\infty(\log t)^\alpha t^{\lambda-1}e^{-\varrho t}\,dt,\qquad(7.11)$$

assuming that α and λ are real, $c>1$, $\varrho>0$ and $\varrho\to 0+$.

In place of (7.3) we have in this case

$$\int_0^c(\log t)^k t^{\lambda-1}e^{-\varrho t}\,dt=O\left(\frac{c^{k-\delta}}{\lambda-\delta}\right)\qquad(7.12)$$

uniformly for $\lambda>\delta>0$ and $\varrho>0$. This result follows upon remarking that $e^{-\varrho t}\leqq 1$ and that given any $\delta>0$, there exists an $A>0$, depending on k and δ and c but not on λ or ϱ, so that $|\log t|^k\leqq A\,t^{-\delta}$ for $0<t\leqq c$.

From (7.2), with the lower signs, and (7.12) we have for any non-negative integer k,

$$M(\varrho,\lambda,k)=\sum_{r=0}^{k}\binom{k}{r}\Gamma^{(r)}(\lambda)\,\varrho^{-\lambda}(-\log\varrho)^{k-r}+O\left((\lambda-\delta)^{-1}c^{\lambda-\delta}\right)\qquad(7.13)$$

uniformly for $\varrho>0$ and $\lambda>\delta>0$.

The extension of this result to arbitrary real values of α will be based on the functional equation

$$\frac{1}{\Gamma(\beta)}\int_0^\infty z^{\beta-1}M(\varrho,\lambda-z,\gamma)\,dz=M(\varrho,\lambda,\gamma-\beta)\qquad\beta>0,\quad\varrho>0,\qquad(7.14)$$

the proof of which is analogous to that of (7.5).

Lemma 4. *For $\lambda>0$, α real, and $\varrho\to 0+$,*

$$M(\varrho,\lambda,\alpha)\sim\sum_{n=0}^{\infty}\binom{\alpha}{n}\Gamma^{(n)}(\lambda)\,\varrho^{-\lambda}(-\log\varrho)^{\alpha-n}\qquad\{\varrho^{-\lambda}(\log\varrho)^{\alpha-n}\}.$$

Proof. As before, we set $c=k-\beta$ where k is a sufficiently large positive integer and $\beta>0$, and we set

$$M(\varrho,\lambda,\alpha)=\frac{1}{\Gamma(\beta)}\left(\int_0^Z+\int_Z^\infty\right)z^{\beta-1}M(\varrho,\lambda-z,k)\,dz=M_1+M_2,$$

say, where $0<Z<\lambda$. As in the proof of (7.7),

$$M_2=O\left(\int_c^\infty(\log t)^k\,t^{\lambda-Z-1}\,e^{-\varrho t}\,dt\right)$$
$$=O\left(\varrho^{-\lambda+Z}(-\log\varrho)^k\right)\qquad\varrho\to 0+$$

by (7.13), and with $0<\delta<\lambda-Z$,

$$\frac{1}{\Gamma(\beta)}\int_0^Z z^{\beta-1}\,O\left((\lambda-\delta-z)^{-1}c^{\lambda-\delta-z}\right)dz=O(1)\qquad\varrho\to 0+.$$

The last two results correspond to (7.7) and (7.8); and from here on the proof follows closely that of lemma 3 and need not be given in full.

For the purposes of the next section, only the first terms of the asymptotic expansions given in lemmas 3 and 4, *i.e.* the results

$$\frac{L(\varrho,\lambda,\alpha)}{\Gamma(\lambda)\,\varrho^{-\lambda}(\log\varrho)^\alpha}\to 1\quad\text{as}\quad\varrho\to+\infty,\tag{7.15}$$

$$\frac{M(\varrho,\lambda,\alpha)}{\Gamma(\lambda)\,\varrho^{-\lambda}(-\log\varrho)^\alpha}\to 1\quad\text{as}\quad\varrho\to+\infty\tag{7.16}$$

are needed. In both results $\lambda>0$ and α is real.

8. Expansions involving logarithms

We first set

$$\psi_n(t)=(-\log t)^{\alpha_n}\,t^{\lambda_n-1}$$
$$\varphi_n(t)=(-\log t)^{\operatorname{Re}\alpha_n}\,t^{\operatorname{Re}\lambda_n-1}\qquad 0<t<c<1,$$
$$=0\qquad\qquad t>c$$

where α_n and λ_n are complex parameters, and $\operatorname{Re}\lambda_n>0$. Under certain assumptions on α_n and λ_n, $\{\psi_n(t)\}$ and $\{\varphi_n(t)\}$ are equivalent asymptotic sequences for $t\to 0+$. Moreover, φ_n is clearly in L, $\varphi_n(t)\geqq 0$ for $t>0$, and $\varphi_n(t)>0$ for $0<t<c$. Thus, $\{\varphi_n(t)\}$ satisfies the conditions of theorem 1 (i), and we may apply theorem 3. $\{\Phi_n(\varrho)\}=\{L(\varrho,\operatorname{Re}\lambda_n,\operatorname{Re}\alpha_n)\}$ is then an asymptotic sequence for $\varrho\to+\infty$, and by (7.15) this asymptotic sequence is equivalent to $\{(\log\varrho)^{\operatorname{Re}\alpha_n}\varrho^{-\operatorname{Re}\lambda_n}\}$ which in its turn is equivalent to $\{(\log\varrho)^{\alpha_n}\varrho^{-\lambda_n}\}$ for $\varrho\to+\infty$. Lastly, one proves exactly as in section 6 that the latter sequence is equivalent to $\{(\log p)^{\alpha_n}p^{-\lambda_n}\}$ provided that p is restricted to the sector S_Δ with some $\Delta>0$. Thus, from theorem 3 the following result is obtained.

Theorem 7. *For each n, let* $\operatorname{Re}\lambda_n > 0$ *and either* $\operatorname{Re}\lambda_n > \operatorname{Re}\lambda_{n-1}$ *and* α_n *arbitrary or* $\operatorname{Re}\lambda_n = \operatorname{Re}\lambda_{n-1}$ *and* $\operatorname{Re}\alpha_n < \operatorname{Re}\alpha_{n-1}$. *If*

$$f(t) \sim \sum f_n(t) \qquad \{(-\log t)^{\alpha_n} t^{\lambda_n - 1}\} \quad \text{as} \quad t \to 0+, \tag{8.1}$$

and if f and all the f_n *are in L; then*

$$F(p) \sim \sum F_n(p) \qquad \{(\log \varrho)^{\alpha_n} \varrho^{-\alpha_n}\} \quad \text{as} \quad p \to \infty \text{ in some } R_0, \tag{8.2}$$

and

$$F(p) \sim \sum F_n(p) \qquad \{(\log p)^{\alpha_n} p^{-\alpha_n}\} \quad \text{as} \quad p \to \infty \text{ in } S_\Delta, \quad \Delta > 0. \tag{8.3}$$

Similarly, we may start with

$$\begin{aligned} \psi_n(t) &= (\log t)^{\alpha_n} t^{\lambda_n - 1}, \\ \varphi_n(t) &= (\log t)^{\operatorname{Re}\alpha_n} t^{\operatorname{Re}\lambda_n - 1} \qquad t > c > 1 \\ &= 0 \qquad\qquad\qquad\qquad\quad 0 < t < c, \end{aligned}$$

where $\operatorname{Re}\lambda_n > 0$, and note that under certain assumptions on α_n and λ_n, $\{\psi_n(t)\}$ and $\{\varphi_n(t)\}$ are equivalent asymptotic sequences for $t \to +\infty$, and $\{\varphi_n(t)\}$ satisfies the conditions of theorem 2. Thus, $\{\Phi_n(\varrho)\}$ is an asymptotic sequence for $p \to 0$ in the half-plane $\varrho > 0$, and (7.16) in connection with an argument similar to that preceding the statement of theorem 5 may be used to replace $\{\Phi_n(\varrho)\}$ by $\{(-\log \varrho)^{\alpha_n} \varrho^{-\lambda_n}\}$ in R_0 and by $\{(-\log p)^{\alpha_n} p^{-\lambda_n}\}$ in S_Δ. Finally one obtains

Theorem 8. *For each n let* $\operatorname{Re}\lambda_n > 0$ *and either* $\operatorname{Re}\lambda_n < \operatorname{Re}\lambda_{n-1}$ *and* α_n *arbitrary or* $\operatorname{Re}\lambda_n = \operatorname{Re}\lambda_{n-1}$ *and* $\operatorname{Re}\alpha_n < \operatorname{Re}\alpha_{n-1}$. *If*

$$f(t) \sim \sum f_n(t) \qquad \{(\log t)^{\alpha_n} t^{\lambda_n - 1}\} \quad \text{as} \quad t \to +\infty, \tag{8.4}$$

and if f and all the f_n *are in* L_0; *then*

$$F(p) \sim \sum F_n(p) \qquad \{(-\log \varrho)^{\alpha_n} \varrho^{-\lambda_n}\} \tag{8.5}$$

as $p \to 0$ *in the half-plane* $\varrho > 0$, *and*

$$F(p) \sim \sum F_n(p) \qquad \{(-\log p)^{\alpha_n} p^{-\lambda_n}\} \tag{8.6}$$

as $p \to 0$ *in* S_Δ, $\Delta > 0$.

These results appear to be new even in the restricted case when (8.1) and (8.4) are asymptotic expansions in the sense of Poincaré. If the α_n are not integers, an interesting situation arises in this case. If we have, say,

$$f(t) \sim \sum a_n (-\log t)^{\alpha_n} t^{\lambda_n - 1} \quad \text{as} \quad t \to 0+$$

in the sense of Poincaré, with constant a_n, we obtain, with an extension of (7.1) to complex α, λ, and p,

$$F(p) \sim \sum a_n L(p, \lambda_n, \alpha_n) \qquad \{(\log \varrho)^{\alpha_n} \varrho^{-\lambda_n}\}$$

as $p \to \infty$ in R_0. Here the terms of the expansion depend on the choice of c used in the construction of L, and in point of fact a different c_n could be used for each n. Thus, in the case of general α_n, the asymptotic expansion of $F(p)$ is not unique, and it is impossible to single out any particular expansion as being simpler or better than the others. Moreover, one has the choice of regarding

the asymptotic expansion of $F(p)$ either as an asymptotic expansion in the sense of Poincaré with respect to the comparatively involved and non-unique asymptotic sequence $\{L(p, \lambda_n, \alpha_n)\}$ for $p \to \infty$ in S_Δ, or else as a general asymptotic expansion with respect to the comparatively simple asymptotic sequence $\{(-\log \varrho)^{\alpha_n} \varrho^{-\lambda_n}\}$ and valid in the larger region R_0. There is little doubt about the second point of view being preferable in most problems.

The situation simplifies considerably if all the α_n are integers so that $(-\log t)^{\alpha_n}$ possesses a unique determination for all $t>0$, and $c=\infty$ may be taken in (7.1) for all n. If, moreover, the α_n are non-negative integers, the expansion (7.2) may be used, and the asymptotic expansion of $F(p)$ in S_Δ becomes either an asymptotic expansion in the sense of Poincaré with respect to the asymptotic sequence

$$\left\{\sum_{r=0}^{\alpha_n} (-1)^r \binom{\alpha_n}{r} \Gamma^{(r)}(\lambda_n)\, p^{-\lambda_n} (\log p)^{\alpha_n - r}\right\}$$

or else, upon rearrangement of the terms, an asymptotic expansion in Poincaré's sense with respect to the sequence $\{p^{-\lambda_n}(-\log p)^{\alpha_n - r}\}$ in which n runs through the same values as in (8.1), for each n, r runs through $0, 1, \ldots, \alpha_n$, and the terms of the sequence have been rearranged, if necessary, in decreasing orders of magnitude as $p \to \infty$, and eliminating duplications.

In the literature known to the present writer, only the case of non-negative integer α_n has been considered, and precise results are known mostly when $\alpha_n = 0$ or 1 [see DOETSCH (1950—1956) vol. 2, p. 45—50].

9. Some Laplace integrals involving exponential functions

For the discussion of asymptotic expansions involving exponential functions we require some information on the behavior of the Laplace transforms of functions like $t^{\lambda-1} \exp(\alpha t^\beta)$ as $p \to 0$ or $p \to \infty$. The results which we require are not essentially new but there does not seem to be a convenient reference giving all of them; and we propose to prove them here, especially since their proof illustrates the application of the results of section 6. In point of fact, we use the results on Laplace integrals to provide a justification of Laplace's method [DOETSCH (1950—1956) vol. 2, p. 83f; ERDÉLYI (1956) sec. 2.4]. For the sake of simplicity we restrict ourselves to real parameters and set

$$h(t, \alpha, \beta, \lambda) = t^{\lambda-1} \exp(\alpha t^\beta) \tag{9.1}$$

$$H(\varrho, \alpha, \beta, \lambda) = \int_0^\infty t^{\lambda-1} \exp(\alpha t^\beta - \varrho t)\, dt, \tag{9.2}$$

the latter provided the integral exists.

We shall use h as a comparison function, and it is necessary to satisfy the conditions of lemmas 1 and 2. In the case of lemma 2, $t \to \infty$. In order to make h differ in its asymptotic behavior from $t^{\lambda-1}$, we must assume $\beta > 0$ and $\alpha \neq 0$; integrability at $t=0$ demands $\lambda > 0$, and the condition that h possess no Laplace transform for $p=0$ clearly demands $\alpha > 0$. Since $\alpha > 0$, h will possess a Laplace transform for $p = \varrho > 0$ only if $\beta < 1$. In the case of lemma 1, $t \to 0+$, and the asymptotic behavior of h will differ from that of $t^{\lambda-1}$ if $\beta < 0$ and $\alpha \neq 0$. Integrability at $t=0$ then demands $\alpha < 0$, and no condition need be imposed on λ.

Accordingly, we shall investigate the asymptotic behavior of $H(\varrho, \alpha, \beta, \lambda)$ under two sets of conditions: (i) $\alpha>0$, $0<\beta<1$, $\lambda>0$, and $\varrho\to 0+$; and (ii) $\alpha<0$, $\beta<0$, λ arbitrary (real), and $\varrho\to\infty$. In either of these cases the integral in (9.2) clearly converges for all $\varrho>0$.

In both cases we introduce the notations

$$\gamma=\frac{1}{1-\beta}, \qquad z=(\alpha\beta\varrho^{-\beta})^{\gamma}, \tag{9.3}$$

note that γ and z are positive, and $z\to+\infty$ in both cases. Let us introduce a new variable of integration x in (9.2) by the substitution $t=(\alpha\beta\varrho^{-1})^{\gamma}x$, obtaining

$$H(\varrho, \alpha, \beta, \lambda)=\left(\frac{\alpha\beta}{\varrho}\right)^{\gamma\lambda}\int_0^\infty x^{\lambda-1}e^{-z(x-\beta^{-1}x^{\beta})}\,dx. \tag{9.4}$$

We wish to investigate the behavior of this integral as $z\to\infty$.

Now, the integral in (9.4) is not a Laplace integral, and so theorem 5 does not apply directly. Nevertheless, it is possible to transform this integral into a Laplace integral by the substitution

$$y=x-\beta^{-1}x^{\beta}-1+\beta^{-1} \tag{9.5}$$

and then apply theorem 5.

The function of x on the right hand side of (9.5) is strictly decreasing when $x<1$ and strictly increasing when $x>1$, and it becomes necessary to break the integral in (9.4) in two parts, and transform each part separately.

The substitution (9.5) maps the interval $1<x<\infty$ onto the interval $0<y<\infty$, and we have

$$\int_1^\infty x^{\lambda-1}e^{-z(x-\beta^{-1}x^{\beta})}\,dx=e^{-(1-\beta^{-1})z}\int_0^\infty f_1(y)\,e^{-yz}\,dy$$

where

$$f_1(y)=x^{\lambda-1}\frac{dx}{dy} \qquad x>1.$$

(9.5) also maps the interval $0<x<1$ onto the interval $\beta^{-1}-1>y>0$ if $0<\beta<1$, and $\infty>y>0$ if $\beta<0$, so that

$$\int_0^1 x^{\lambda-1}e^{-z(x-\beta^{-1}x^{\beta})}\,dx=-e^{-(1-\beta^{-1})z}\int_0^\infty f_2(y)\,e^{-yz}\,dy,$$

where

$$f_2(y)=x^{\lambda-1}\frac{dx}{dy} \qquad 0<x<1,$$

and $f_2(y)=0$ for $y>\beta^{-1}-1$ in case (i).

Setting $f(y)=f_1(y)-f_2(y)$, we thus obtain

$$H(\varrho, \alpha, \beta, \lambda)=\left(\frac{\alpha\beta}{\varrho}\right)^{\gamma\lambda}e^{-(1-\beta^{-1})z}\int_0^\infty f(y)\,e^{-yz}\,dy, \tag{9.6}$$

a representation of a multiple of H by a Laplace integral in which $z\to+\infty$ in both cases (i) and (ii). It is clear from (9.4) that f is in L, and it follows from

theorem 5 that an asymptotic expansion of f in series of ascending powers of y will lead to an asymptotic expansion of the Laplace integral in descending powers of z. Only the leading term of this asymptotic expansion will be computed here explicitly.

Lemma 5. *If either* (i) $\alpha>0$, $0<\beta<1$, $\lambda>0$ *and* $\varrho\to 0+$ *or* (ii) $\alpha<0$, $\beta<0$, λ *arbitrary, and* $\varrho\to+\infty$, *then*

$$H(\varrho,\alpha,\beta,\lambda)\sim\left(\frac{\alpha\beta}{\varrho}\right)^{\gamma\lambda}e^{-(1-\beta^{-1})z}\sum_{n=0}^{\infty}\Gamma\left(\frac{n+1}{2}\right)c_n\,z^{-(n+1)/2}, \tag{9.7}$$

where γ, z *are defined in* (9.3), *the* c_n *are independent of* z, *and depend only on* β *and* λ, $c_0=\sqrt{2\gamma}$, *and* (9.7) *is an asymptotic expansion in Poincaré's sense. In particular*

$$\frac{H(\varrho,\alpha,\beta,\lambda)}{\sqrt{2\pi\gamma}\,(\alpha\beta)^{(\lambda-\frac{1}{2})\gamma}\,\varrho^{(\frac{1}{2}\beta-\lambda)\gamma}\exp-(1-\beta^{-1})\,z}\to 1 \tag{9.8}$$

under either of the conditions (i) *or* (ii).

Proof. The right hand side of (9.5) is regular at $x=1$, the point corresponding to $y=0$. Using $\mathfrak{P}(w)$ as a generic symbol for a power series in w such that $\mathfrak{P}(0)=0$, we have by explicit computation

$$y=\frac{1-\beta}{2}(x-1)^2[1+\mathfrak{P}(x-1)] \qquad |x-1|<1.$$

By inversion of power series we have, in some neighborhood of $y=0$

$$\pm(x-1)=\sqrt{2\gamma\,y}\,[1+\mathfrak{P}(\sqrt{y})],$$

where $\sqrt{y}$ denotes the positive square root, the upper sign is to be used for $x>1$ (*i.e.*, in f_1) and the lower sign, for $0<x<1$ (*i.e.*, in f_2). Hence

$$\pm\frac{dx}{dy}=\sqrt{\frac{\gamma}{2y}}\,[1+\mathfrak{P}(\sqrt{y})],$$

and since

$$x^{\lambda-1}=1+\mathfrak{P}(x-1)=1+\mathfrak{P}(\sqrt{y}),$$

we finally obtain

$$f(y)=\sqrt{\frac{2\gamma}{y}}\,[1+\mathfrak{P}(\sqrt{y})]=\sum_{n=0}^{\infty}c_n\,y^{(n-1)/2},$$

say, where $c_0=\sqrt{2\gamma}$, and the power series converges, and represents $f(y)$, in some punctured neighborhood of $y=0$.

Under these circumstances $\sum c_n\,y^{(n-1)/2}$ represents $f(y)$ asymptotically, in Poincaré's sense, as $y\to 0+$, and it follows from theorem 5, or (6.5), that

$$\int_0^{\infty}f(y)\,e^{-yz}\,dy\sim\sum_{n=0}^{\infty}c_n\,\Gamma\left(\frac{n+1}{2}\right)z^{-(n+1)/2} \quad\text{as}\quad z\to+\infty,$$

again in Poincaré's sense. Substitution in (9.6) leads to (9.7), and (9.8) is a simple consequence of (9.7).

It may be mentioned that certain well-known asymptotic expansions of modified Bessel functions and of parabolic cylinder functions are special instances

(for $\beta=-1$ and $\beta=\frac{1}{2}$ respectively) of (9.7); and that the validity of (9.7) can be extended easily to complex values of the parameters. In particular, (9.7) is valid for complex λ with $\operatorname{Re}\lambda>0$ in case (i), and no restriction on λ in case (ii).

10. Expansions involving exponential functions

By taking appropriate sequences of α, β, λ in (9.1) we can now construct asymptotic sequences, in case (i) for $t\to\infty$ and in case (ii) for $t\to 0$. It follows from lemma 5 that these sequences satisfy the conditions of theorems 1 and 2, since h is clearly in L_0' in case (i) of lemma 5, and in L in case (ii), and $h(t)>0$ for $t>0$. Thus, the Laplace transforms also form an asymptotic sequence. By (9.8), the latter sequence is equivalent to the corresponding sequence of functions

$$X(\varrho)=\varrho^{(\frac{1}{2}\beta-\lambda)\gamma}\, e^{(\beta^{-1}-1)z}$$

as $\varrho\to 0+$ in case (i) and $\varrho\to+\infty$ in case (ii). An examination of the exponential part shows that for complex $p\to 0$ in case (i), and $\to\infty$ in case (ii), $X(\varrho)/X(p)$ cannot remain bounded unless* $\arg p\to 0$ so that there is no useful region R in the complex plane in which the sequence of the $X(p)$ is equivalent to the sequence of the $X(\varrho)$.

For the sake of simplicity, we restrict α and β to real values, allowing λ to be complex.

In case (i), we set

$$\begin{aligned}\psi_n(t)&=t^{\lambda_n-1}\exp(\alpha_n t^{\beta_n}),\\ \gamma_n&=(1-\beta_n)^{-1},\\ X_n(\varrho)&=\varrho^{-(\lambda_n-\frac{1}{2}\beta_n)\gamma_n}\exp\left[(\beta_n^{-1}-1)(\alpha_n\beta_n\varrho^{-\beta_n})^{\gamma_n}\right],\end{aligned}\tag{10.1}$$

and verify that under the conditions given in theorem 9, $\{\psi_n(t)\}$ is an asymptotic sequence as $t\to\infty$, $\{X_n(\varrho)\}$ is an asymptotic sequence as $\varrho\to 0+$, and the conditions of theorem 4 are met. Thus we obtain

Theorem 9. *For each n let* (i) $\alpha_n>0$, $0<\beta_n<1$, $\operatorname{Re}\lambda_n>0$; *and either* (ii) $\beta_n<\beta_{n-1}$, *or* (iii) $\beta_n=\beta_{n-1}$ *and* $\alpha_n<\alpha_{n-1}$, *or else* (iv) $\beta_n=\beta_{n-1}$, $\alpha_n=\alpha_{n-1}$, *and* $\operatorname{Re}\lambda_n<\operatorname{Re}\lambda_{n-1}$. *If*

$$f(t)\sim\sum f_n(t)\qquad\{\psi_n(t)\}\tag{10.2}$$

as $t\to+\infty$, where ψ_n is given in (10.1), *and if f and all the f_n are in L_0; then*

$$F(p)\sim\sum F_n(p)\qquad\{X_n(\varrho)\}\tag{10.3}$$

as $p\to 0$ in the half-plane $\varrho>0$, X_n being defined in (10.1).

In case (ii) we change the notation slightly, setting

$$\begin{aligned}\psi_n(t)&=t^{\lambda_n-1}\exp(-\alpha_n t^{-\beta_n})\\ \gamma_n&=(1+\beta_n)^{-1}\\ X_n(\varrho)&=\varrho^{-(\lambda_n+\frac{1}{2}\beta_n)\gamma_n}\exp\left[-(\beta_n^{-1}+1)(\alpha_n\beta_n\varrho^{\beta_n})^{\gamma_n}\right].\end{aligned}\tag{10.4}$$

* In a special case this examination was carried out by Doetsch (1950—1956), vol. 2, p. 94.

Under the conditions set out in theorem 10, $\{\psi_n\}$ is an asymptotic sequence as $t\to 0+$, $\{X_n\}$ is an asymptotic sequence as $\varrho\to\infty$, and the conditions of theorem 3 are met. We then have

Theorem 10. *For each n let* (i) $\alpha_n>0$, $\beta_n>0$; *and either* (ii) $\beta_n>\beta_{n-1}$, *or* (iii) $\beta_n=\beta_{n-1}$ *and* $\alpha_n>\alpha_{n-1}$, *or else* (iv) $\beta_n=\beta_{n-1}$, $\alpha_n=\alpha_{n-1}$, *and* $\operatorname{Re}\lambda_n>\operatorname{Re}\lambda_{n-1}$. *If, under these circumstances, and with ψ_n given in* (10.4), (10.2) *holds as $t\to 0+$, and if f and all the f_n are in L; then* (10.3) *holds, with X_n as in* (10.4), *as $p\to\infty$ in some R_0.*

This is a straightforward deduction from theorem 3. Since for each n and each $\varepsilon>0$, $X_n(\varrho)$ is bounded below by some multiple of $e^{-\frac{1}{2}\varepsilon\varrho}$, condition (5.1) is clearly satisfied in R_0.

The writing of this paper was sponsored by the United States Army under Contract No. DA-11-022-ORD-2059, Mathematics Research Center, United States Army, Madison, Wisconsin.

References

VAN DER CORPUT, J. G.: Asymptotic developments, I. Fundamental theorems of asymptotics. J. Analyse Math. **4**, 341—418 (1955/56).

DOETSCH, GUSTAV: Handbuch der Laplace-Transformation, 3 vols. Basel: Birkhäuser 1950—56.

ERDÉLYI, A.: Asymptotic representation of Laplace transforms with an application to inverse factorial series. Proc. Edinburgh Math. Soc. (2) **8**, 20—24 (1947).

ERDÉLYI, A.: Asymptotic Expansions. Dover 1956.

KAPLUN, S., & P. A. LAGERSTROM: Asymptotic expansions of Navier-Stokes solutions for small Reynolds numbers. J. Math. Mechanics **6**, 585—593 (1957).

SCHMIDT, HERMANN: Beiträge zu einer Theorie der allgemeinen asymptotischen Darstellungen. Math. Ann. **113**, 629—656 (1936).

WIDDER, D. V.: The Laplace Transform. Princeton 1941.

WYMAN, MAX: Asymptotic expansion of coefficients of Laurent series. J. Soc. Ind. and Applied Math. **7**, 325—342 (1959).

California Institute of Technology
Pasadena, California

(Received October 21, 1960)

Über den Resonanzbegriff bei Systemen von n linearen gewöhnlichen Differentialgleichungen erster Ordnung

RAHMI IBRAHIM IBRAHIM ABDEL KARIM

Communicated by L. COLLATZ

§ 1. Problemstellung, Hauptergebnisse

R. IGLISCH untersuchte den Resonanzbegriff bei linearen gewöhnlichen Differentialgleichungen zweiter Ordnung (Arch. Rational Mech. Anal. **3**, 179–186 (1959)). Diese Betrachtungen sollen ausgedehnt werden auf Systeme von n gewöhnlichen linearen Differentialgleichungen erster Ordnung, die in Matrixform mit $\mathfrak{x}(t)$ als gesuchtem Vektor geschrieben werden sollen:

$$\frac{d\mathfrak{x}}{dt} = \mathfrak{A}(t)\,\mathfrak{x} + \mathfrak{f}(t); \tag{1}$$

dabei soll die quadratische (z.B. reelle) Matrix $\mathfrak{A}(t)$ etwa stetig in t und periodisch mit der Periode P sein, d.h. alle n^2 Elemente $a_{ik}(t)$ sind stetige mit P periodische Funktionen in t; der (z.B. auch reelle) Vektor $\mathfrak{f}(t)$ sei etwa gleichfalls stetig und mit P periodisch:

$$\mathfrak{A}(t+P) = \mathfrak{A}(t); \qquad \mathfrak{f}(t+P) = \mathfrak{f}(t). \tag{2}$$

Das zu (1) gehörende homogene System ist

$$\frac{d\mathfrak{y}}{dt} = \mathfrak{A}(t)\,\mathfrak{y} \tag{3}$$

und das „adjungierte“

$$\frac{d\mathfrak{z}}{dt} = -\mathfrak{A}^T(t)\,\mathfrak{z}, \tag{4}$$

wobei durch das hochgestellte T der Übergang zur transponierten Matrix gekennzeichnet sei.

Definition 1. Bei dem inhomogenen Differentialgleichungssystem (1) liegt der *Resonanzfall* vor, wenn das adjungierte System (4) mindestens einen mit P periodischen Lösungsvektor $\mathfrak{z}(t)$ besitzt, für den

$$\int_0^P \mathfrak{z}^T(t)\,\mathfrak{f}(t)\,dt = C \neq 0 \tag{5}$$

ist.

In §4 wird bewiesen:

Satz 1. *Im Resonanzfall nimmt jeder Lösungsvektor $\mathfrak{x}(t)$ von (1) mit unbeschränkt wachsendem t beliebig große Beträge an.*

Definition 2. Bei (1) liegt der Hauptfall vor, wenn (4) keinen mit P periodischen Lösungsvektor $\mathfrak{z}(t)$ besitzt.

In §5 ergibt sich:

Satz 2. *Im Hauptfall gibt es für alle t beschränkt bleibende Lösungen $\mathfrak{x}(t)$ von (1), z.B. die eindeutig existierende mit P periodische Lösung.*

Definition 3. Bei (1) liegt der *Ausnahmefall* vor, wenn zwar (4) mit P periodische Lösungsvektoren $\mathfrak{z}(t)$ besitzt, jedoch für alle diese mit P periodischen Lösungen gilt:

$$\int_0^P \mathfrak{z}^T(t)\,\mathfrak{f}(t)\,dt = 0. \tag{6}$$

In §6 wird gezeigt:

Satz 3. *Auch im Ausnahmefall gibt es für alle Werte von t beschränkt bleibende Lösungen $\mathfrak{x}(t)$ von (1), z.B. die mit P periodischen, die aber jetzt nicht mehr eindeutig bestimmt sind.*

Die §§ 2 und 3 sind Hilfsbetrachtungen über das inhomogene System (1), vor allem aber über den Zusammenhang zwischen dem homogenen System (3) und dem adjungierten System (4) vorbehalten. Dabei wird die Periodizitätsvoraussetzung (2) erst in §3 eingeführt.

Viele Ergebnisse sind bekannt, werden aber hier auf eine Weise abgeleitet, die keine besonderen Vorkenntnisse verlangt.

§2. Hilfsbetrachtungen über Systeme linearer Differentialgleichungen erster Ordnung

In diesem Paragraphen wird die Periodizitätsvoraussetzung (2) nicht benutzt.

$\mathfrak{y}_1(t), \ldots, \mathfrak{y}_n(t)$ sei an der Stelle t_0 ein linear unabhängiges Lösungssystem (Fundamentalsystem) von (3), das wir zu der Lösungsmatrix

$$\mathfrak{Y}(t) = \big(\mathfrak{y}_1(t), \ldots, \mathfrak{y}_n(t)\big) \tag{7}$$

zusammenfassen wollen. Dann ist

$$\operatorname{Det} \mathfrak{Y}(t_0) \neq 0. \tag{8}$$

Satz 4. *Aus* (8) *folgt*

$$Y(t) = \operatorname{Det} \mathfrak{Y}(t) \neq 0 \tag{9}$$

für alle Werte von t.

Beweis. Bezeichnet man die Adjunkten (Unterdeterminanten mit richtigem Vorzeichen $(-1)^{i+\nu}$) zum Element $y_{i\nu}(t)$ in (7) oder (9) mit $Y_{i\nu}(t)$ und setzt

$$\delta_{ik} = \begin{cases} 0 & \text{für} \quad i \neq k, \\ 1 & \text{für} \quad i = k, \end{cases} \tag{10}$$

so rechnet sich unter Beachtung von (3), wenn die Differentiation nach t durch einen Strich gekennzeichnet wird,

$$\frac{dY}{dt} = \sum_{\nu,i} y'_{i\nu} Y_{i\nu} = \sum_{\nu,i,k} a_{ik}\, y_{k\nu} Y_{i\nu} = \sum_{i,k} a_{ik}\, \delta_{ki}\, Y = Y \sum_i a_{ii}$$

und somit

$$Y(t) = Y(t_0) \exp\left(\int_{t_0}^{t} (a_{11} + a_{22} + \cdots + a_{nn})\, dt \right).$$

Hieraus folgt unmittelbar Satz 4.

Satz 5. *Neben (7) ist auch*

$$\mathfrak{Y}^*(t) = \mathfrak{Y}(t) \cdot \mathfrak{C} \tag{11}$$

bei beliebiger konstanter Matrix $\mathfrak{C}$ mit von Null verschiedener Determinante C ein Fundamentalsystem von Lösungen von (3).

Beweis. (7) und (3) kann man mit

$$\mathfrak{Y}' = (\mathfrak{y}_1', \mathfrak{y}_2', \ldots, \mathfrak{y}_n') \tag{12}$$

zusammenfassen zu

$$\frac{d\mathfrak{Y}}{dt} = \mathfrak{A}(t)\, \mathfrak{Y}. \tag{13}$$

Damit rechnet sich

$$(\mathfrak{Y}\mathfrak{C})' = \mathfrak{Y}'\mathfrak{C} = \mathfrak{A}\,\mathfrak{Y}\mathfrak{C} = \mathfrak{A}(\mathfrak{Y}\mathfrak{C}).$$

Jedes $\mathfrak{Y}\mathfrak{C}$ ist also Lösungssystem von (3).

Daß außerdem $\det(\mathfrak{Y}\mathfrak{C}) \neq 0$ ist, folgt aus

$$\det(\mathfrak{Y}\mathfrak{C}) = Y \cdot C \neq 0.$$

Satz 6. *Die Vektoren*

$$\mathfrak{z}_i(t) = \frac{1}{Y(t)} \begin{pmatrix} Y_{1i}(t) \\ Y_{2i}(t) \\ \vdots \\ Y_{ni}(t) \end{pmatrix}, \qquad i = 1, \ldots, n, \tag{14}$$

die man in die Matrix

$$\mathfrak{Z}(t) = \left(\frac{1}{Y(t)} \cdot Y_{ki}(t) \right) \tag{15}$$

zusammenfassen kann, bilden ein Fundamentalsystem von Lösungen von (4).

Beweis. Zunächst sieht man sofort an (15), daß

$$\mathfrak{Z}^T \cdot \mathfrak{Y} = \mathfrak{E}, \tag{16}$$

d.h. die Einheitsmatrix ist. Damit kann man unter Einführung der reziproken Matrix $\mathfrak{Y}^{-1}$ auch schreiben

$$\mathfrak{Z}^T = \mathfrak{Y}^{-1}, \qquad \mathfrak{Z}^T = (\mathfrak{Y}^{-1})^T. \tag{17}$$

Aus (16) oder (17) folgt unmittelbar:

$$Z = \det \mathfrak{Z} \neq 0.$$

Daß $\mathfrak{Z}$ Lösungsmatrix von (4) ist, folgt so:

Gemäß (16) ist

$$\mathfrak{Z}^{T\prime}\, \mathfrak{Y} + \mathfrak{Z}^T\, \mathfrak{Y}' = 0,$$

also mit (13)

$$-\mathfrak{Z}^{T\prime}\mathfrak{Y}=\mathfrak{Z}^T\mathfrak{A}\mathfrak{Y}, \quad -\mathfrak{Z}^{T\prime}=\mathfrak{Z}^T\mathfrak{A}\mathfrak{Y}\mathfrak{Y}^{-1}=\mathfrak{Z}^T\mathfrak{A},$$

mithin

$$\mathfrak{Z}'=-\mathfrak{A}^T\mathfrak{Z}, \tag{18}$$

was mit (4) übereinstimmt.

Aus den Sätzen (5) und (6) folgt nun sofort

Satz 7. *Allgemeines Fundamentalsystem von Lösungen von (4) ist die Matrix*

$$\mathfrak{Z}^*=\mathfrak{Z}\cdot\mathfrak{C}=(\mathfrak{Y}^{-1})^T\cdot\mathfrak{C} \tag{19}$$

bei beliebiger konstanter Matrix $\mathfrak{C}$ mit von Null verschiedener Determinante C.

Bezeichnet man

$$L(\mathfrak{y})\equiv\frac{d\mathfrak{y}}{dt}-\mathfrak{A}(t)\,\mathfrak{y} \tag{20}$$

und

$$L^*(\mathfrak{z})\equiv-\frac{d\mathfrak{z}}{dt}-\mathfrak{A}^T(t)\,\mathfrak{z}, \tag{21}$$

so gilt für zwei beliebige Vektoren $\mathfrak{y}(t)$ und $\mathfrak{z}(t)$

$$\mathfrak{z}^T L(\mathfrak{y})-\big(L^*(\mathfrak{z})\big)^T\cdot\mathfrak{y}=\frac{d}{dt}(\mathfrak{z}^T\cdot\mathfrak{y}) \tag{22}$$

und

$$\big(L(\mathfrak{y})\big)^T\cdot\mathfrak{z}-\mathfrak{y}^T L^*(\mathfrak{z})=\frac{d}{dt}(\mathfrak{y}^T\cdot\mathfrak{z}). \tag{23}$$

Daher gilt die (erweiterte) Lagrangesche Identität

$$\frac{d}{dt}(\mathfrak{z}^T\cdot\mathfrak{y})=0=\frac{d}{dt}(\mathfrak{y}^T\cdot\mathfrak{z}), \tag{24}$$

für Lösungsvektoren $\mathfrak{y}$ und $\mathfrak{z}$ von (3), bzw. (4).

Analog gilt für Lösungsvektoren von (1) und (4)

$$\frac{d}{dt}(\mathfrak{z}^T\cdot\mathfrak{x})=\mathfrak{z}^T\mathfrak{f}, \quad \frac{d}{dt}(\mathfrak{x}^T\cdot\mathfrak{z})=\mathfrak{f}^T\cdot\mathfrak{z}. \tag{25}$$

§ 3. Über Systeme linearer Differentialgleichungen mit periodischen Koeffizienten

In diesem Paragraphen ist die Periodizitätsvoraussetzung (2) wesentlich.

Satz 8. *Die homogenen Systeme (3) und (4) haben die gleiche Anzahl ϱ linear unabhängiger mit P periodischer Lösungen $(0\leqq\varrho\leqq n)$.*

Beweis[1]. ϱ sei die Anzahl der mit P periodischen linear unabhängigen Lösungen von (3). Es braucht nur gezeigt zu werden, daß (4) genau ϱ linear unabhängige Lösungen besitzt, die mit P periodisch sind. Denn der Schluß läßt sich umkehren, da (3) das adjungierte System zu (4) ist.

Fall 1. Es sei $\varrho=n$. Die ganze Matrix $\mathfrak{Y}(t)$ ist hier mit P periodisch, daher auch die Matrix $\mathfrak{Z}(t)$ nach (15).

[1] Von R. IGLISCH.

Fall 2. Es sei $0<\varrho<n$. Seien die $\mathfrak{y}_\nu(t)$ mit $\nu=1, 2, \ldots, \varrho$ mit P periodisch. Dann gilt für jede beliebige Stelle t_1

$$\mathfrak{y}_\mu(t)\Big|_{t_1}^{t_1+P} \equiv \mathfrak{y}_\mu(t_1+P) - \mathfrak{y}_\mu(t_1) \neq 0 \quad \text{für} \quad \mu=\varrho+1, \ldots, n. \tag{26}$$

Stände nämlich in (26) das Gleichheitszeichen, so würde aus (3) unter Beachtung von (2) die Periodizität dieses $\mathfrak{y}_\mu(t)$ mit der Periode P folgen.

Im folgenden sei t_1 eine beliebig gewählte, dann aber festgehaltene Stelle. Integration der ersten Gleichung (24) zwischen t_1 und t_1+P liefert für jedes $k=1, 2, \ldots, n$

$$\mathfrak{z}_k^T(t)\Big|_{t_1}^{t_1+P} \mathfrak{y}_\nu(t_1) = 0 \quad \text{für} \quad \nu=1, 2, \ldots, \varrho. \tag{27}$$

Das sind n lineare homogene Gleichungen, die mindestens die ϱ linear unabhängigen Lösungen $\mathfrak{y}_\nu(t_1)$ für $\nu=1, 2, \ldots, \varrho$ besitzen. Es soll jetzt gezeigt werden, daß kein weiterer davon linear unabhängiger Lösungsvektor $\mathfrak{u}_1$ existieren kann. Sei im Gegenteil angenommen, daß für ein solches $\mathfrak{u}_1$ gleichfalls

$$\mathfrak{z}_k^T(t)\Big|_{t_1}^{t_1+P} \mathfrak{u}_1 = 0 \tag{28}$$

gilt. Dann bestimme man einen Vektor $\mathfrak{u}(t)$ aus (3)

$$\frac{d\mathfrak{u}}{dt} = \mathfrak{A}(t)\,\mathfrak{u} \quad \text{mit} \quad \mathfrak{u}(t_1) = \mathfrak{u}_1. \tag{29}$$

Für diesen Vektor $\mathfrak{u}(t)$ folgt aus der ersten Gleichung (24) durch Integration zwischen t_1 und t_1+P für $k=1, 2, \ldots, n$

$$\mathfrak{z}_k^T(t_1+P)\,\mathfrak{u}(t_1+P) - \mathfrak{z}_k^T(t_1)\,\mathfrak{u}(t_1) = 0$$

und nach Subtraktion von (28)

$$\mathfrak{z}_k^T(t_1+P)\left[\mathfrak{u}(t_1+P) - \mathfrak{u}(t_1)\right] = 0.$$

Das ist ein lineares homogenes Gleichungssystem mit von Null verschiedener Determinante; es muß also

$$\mathfrak{u}(t_1+P) = \mathfrak{u}(t_1)$$

sein, und daraus folgt vermöge (29) und (2) die Periodizität von $\mathfrak{u}(t)$ mit P. Wegen (26) ist also $\mathfrak{u}_1=\mathfrak{u}(t_1)$ von $\mathfrak{y}_1(t_1), \ldots, \mathfrak{y}_\varrho(t_1)$ linear abhängig.

Nun wissen wir also, daß das Gleichungssystem (27) genau ϱ linear unabhängige Lösungen besitzt. Daher hat die Matrix

$$\mathfrak{Z}(t)\Big|_{t_1}^{t_1+P} = \left(\mathfrak{z}_1(t)\Big|_{t_1}^{t_1+P}, \mathfrak{z}_2(t)\Big|_{t_1}^{t_1+P}, \ldots, \mathfrak{z}_n(t)\Big|_{t_1}^{t_1+P}\right)$$

den Rang $n-\varrho$. Durch entsprechende Numerierung läßt sich erreichen, daß gerade $\mathfrak{z}_1(t)\Big|_{t_1}^{t_1+P}, \ldots, \mathfrak{z}_{n-\varrho}(t)\Big|_{t_1}^{t_1+P}$ linear unabhängig sind. Die übrigen dieser Vektoren kann man durch diese linear ausdrücken:

$$\mathfrak{z}_\mu(t)\Big|_{t_1}^{t_1+P} = \sum_{\sigma=1}^{n-\varrho} c_{\mu,\sigma}\,\mathfrak{z}_\sigma(t)\Big|_{t_1}^{t_1+P} \quad \text{für} \quad \mu=n-\varrho+1, \ldots, n. \tag{30}$$

Nun betrachte man die Vektoren

$$\mathfrak{z}_m^*(t) = \begin{cases} \mathfrak{z}_m(t) & \text{für} \quad m=1, 2, \ldots, n-\varrho \\ \mathfrak{z}_m(t) - \sum\limits_{\sigma=1}^{n-\varrho} c_{m,\sigma}\,\mathfrak{z}_\sigma(t) & \text{für} \quad m=n-\varrho+1, \ldots, n. \end{cases} \tag{31}$$

Da

$$\operatorname{Det}\left(\mathfrak{z}_1^*(t), \ldots, \mathfrak{z}_n^*(t)\right) = \operatorname{Det}\left(\mathfrak{z}_1(t), \ldots, \mathfrak{z}_n(t)\right) \neq 0$$

ist, sind die n Vektoren (31) linear unabhängige Lösungen von (4) für alle t (vgl. dazu auch Satz 4). Die $\mathfrak{z}_\nu^*(t)$ mit $\nu = 1, 2, \ldots, n-\varrho$ sind wegen der linearen Unabhängigkeit der Vektoren

$$\mathfrak{z}_\nu^*(t)\Big|_{t_1}^{t_1+P}$$

einschließlich ihrer Linearkombinationen nicht mit P periodisch. Dagegen besitzen die $\mathfrak{z}_\mu^*(t)$ mit $\mu = n-\varrho+1, \ldots, n$ gerade die Periode P, wie aus (30), (4) und (2) abzulesen ist. Damit ist unser Beweis im Fall 2 erbracht.

Fall 3. $\varrho = 0$. Hätte Gleichung (4) eine mit P periodische Lösung, so müßte nach dem Früheren auch Gleichung (3) eine solche besitzen.

§ 4. Der Resonanzfall

Beweis von Satz 1. Es sei also $\mathfrak{z}(t)$ ein mit P periodischer Lösungsvektor von (4), für den (5) gilt.

Wir machen im Gegensatz zur Behauptung die

$$\text{(32)} \qquad \textit{Annahme:} \quad |\mathfrak{x}(t)| \leqq E$$

für alle t und eine beliebig herausgegriffene Lösung $\mathfrak{x}(t)$ von (1). Integration der ersten Gleichung (25) zwischen t und $t + mP$ mit willkürlichem positiv ganzem m liefert unter Beachtung von (5)

$$\text{(33)} \qquad \mathfrak{z}^T(t)\left[\mathfrak{x}(t+mP) - \mathfrak{x}(t)\right] = m \cdot |\mathfrak{C}|.$$

Infolge der Periodizität von $\mathfrak{z}^T(t)$ gilt für alle t eine Beschränkung der Form

$$\text{(34)} \qquad |\mathfrak{z}^T(t)| \leqq D.$$

Damit ergibt sich aus (33) eine Abschätzung

$$D \cdot 2E \geqq m|C|, \qquad E \geqq \frac{m|C|}{2D}.$$

Das liefert bei genügend großem m einen Widerspruch zu (32).

Man kann das Ergebnis auch so aussprechen:

Satz 9. *Im Resonanzfall gibt es für jeden Lösungsvektor $\mathfrak{x}(t)$ von (1) in jedem Intervall*

$$\text{(35)} \qquad t_0 \leqq t \leqq t_0 + mP$$

mindestens eine Stelle t^*, für die

$$\text{(36)} \qquad \mathfrak{x}(t^*) \geqq \frac{m|C|}{2D}$$

ausfällt mit C aus (5) und D aus (34).

§ 5. Der Hauptfall

In diesem Paragraphen wird vorausgesetzt, daß Gleichung (4) keine mit P periodische Lösung $\mathfrak{z}(t)$ besitzt. Zunächst beweisen wir

Satz 10. *Sind $\mathfrak{z}_1(t), \ldots, \mathfrak{z}_n(t)$ linear unabhängige Lösungen von* (4), *so ist*

$$D(t) = \operatorname{Det} \begin{pmatrix} \mathfrak{z}_1^T(\tau)\big|_t^{t+P} \\ \mathfrak{z}_2^T(\tau)\big|_t^{t+P} \\ \vdots \\ \mathfrak{z}_n^T(\tau)\big|_t^{t+P} \end{pmatrix} \neq 0 \quad \textit{für alle } t. \tag{37}$$

Beweis. Wäre für einen speziellen Wert $t = t_2$ im Gegensatz zur Behauptung $D(t_2) = 0$, so könnte man n Zahlen $\alpha_1, \alpha_2, \ldots, \alpha_n$ mit $\alpha_1^2 + \cdots + \alpha_n^2 > 0$ so bestimmen, daß

$$\sum_{\nu=1}^{n} \alpha_\nu \mathfrak{z}_\nu^T(t_2 + P) = \sum_{\nu=1}^{n} \alpha_\nu \mathfrak{z}_\nu^T(t_2)$$

ist. Setzt man

$$\mathfrak{z}_0(t) = \sum_{\nu=1}^{n} \alpha_\nu \mathfrak{z}_\nu(t),$$

so wäre

$$\mathfrak{z}_0(t_2 + P) = \mathfrak{z}_0(t_2).$$

Hieraus würde aber nach (4) und (2)

$$\mathfrak{z}_0(t + P) = \mathfrak{z}_0(t)$$

für alle t folgen im Widerspruch zur Voraussetzung.

Beweis von Satz 2. Wäre $\mathfrak{x}(t)$ eine mit P periodische Lösung von (1), so würde Integration der ersten Gleichung (25) zwischen einer festen Stelle t_3 und $t_3 + P$ liefern:

$$\mathfrak{z}_\nu^T(t)\big|_{t_3}^{t_3+P} \mathfrak{x}(t_3) = \int_{t_3}^{t_3+P} \mathfrak{z}_\nu^T(t)\, \mathfrak{f}(t)\, dt \quad \text{für} \quad \nu = 1, 2, \ldots, n. \tag{38}$$

Da die Koeffizientendeterminante dieses Gleichungssystems nach Satz 10 von Null verschieden ist, läßt sich hieraus $\mathfrak{x}(t_3)$ eindeutig bestimmen. Wir denken uns nun (1) mit diesen Anfangswerten an der Stelle t_3 gelöst und erhalten so einen Lösungsvektor $\mathfrak{x}(t)$. Wir brauchen nur noch zu beweisen: Dieses $\mathfrak{x}(t)$ ist mit P periodisch. Dazu ist in Hinblick auf (2) nur nötig, festzustellen, daß von selbst

$$\mathfrak{x}(t_3 + P) = \mathfrak{x}(t_3) \tag{39}$$

ist. Integration der ersten Gleichung (25) zwischen t_3 und $t_3 + P$ liefert für $\nu = 1, 2, \ldots, n$

$$\mathfrak{z}_\nu^T(t_3 + P) \cdot \mathfrak{x}(t_3 + P) - \mathfrak{z}_\nu^T(t_3) \cdot \mathfrak{x}(t_3) = \int_{t_3}^{t_3+P} \mathfrak{z}_\nu^T(t)\, \mathfrak{f}(t)\, dt.$$

Zieht man hiervon (38) ab, so erscheint das lineare Gleichungssystem

$$\mathfrak{z}_\nu^T(t_3 + P)\left[\mathfrak{x}(t_3 + P) - \mathfrak{x}(t_3)\right] = 0 \tag{40}$$

mit von Null verschiedener Determinante analog zu (8). Also folgt das Verschwinden der eckigen Klammer und damit (39).

§ 6. Der Ausnahmefall

Hat (4) genau ϱ linear unabhängige mit P periodische Lösungen $\mathfrak{z}_1(t), \ldots, \mathfrak{z}_\varrho(t)$, für die

$$\int_t^{t+P} \mathfrak{z}_r^T(t)\, \mathfrak{f}(t)\, dt = 0, \qquad r = 1, 2, \ldots, \varrho \tag{41}$$

gilt, so hat die Determinante (37) genau den Rang $n-\varrho$. Von dem Gleichungssystem (38) sind die ersten ϱ Gleichungen (für $\nu = 1, 2, \ldots, \varrho$) von selbst erfüllt, da nur Nullen darin vorkommen. Aus den restlichen Gleichungen (38) mit $\nu = \varrho + 1, \ldots, n$ kann man ϱ linear unabhängige Vektoren

$$\mathfrak{x}_1^*(t_3), \mathfrak{x}_2^*(t_3), \ldots, \mathfrak{x}_\varrho^*(t_3) \tag{42}$$

bestimmen, dazu durch Lösung von (1) mit den entsprechenden Anfangswerten (42)

$$\mathfrak{x}_1^*(t), \mathfrak{x}_2^*(t), \ldots, \mathfrak{x}_\varrho^*(t). \tag{43}$$

Daß für jedes dieser $\mathfrak{x}_\mu^*(t)$ ($\mu = 1, 2, \ldots, \varrho$) von selbst

$$\mathfrak{x}_\mu^*(t_3 + P) = \mathfrak{x}_\mu^*(t_3) \tag{44}$$

wird, folgt aus dem analog (40) abzuleitenden Gleichungssystem

$$\mathfrak{z}_\nu^T(t_3 + P)\left[\mathfrak{x}_\mu^*(t_3 + P) - \mathfrak{x}_\mu^*(t_3)\right] = 0 \tag{45}$$

mit von Null verschiedener Determinante. Alle diese $\mathfrak{x}_\mu^*(t)$ haben also die Periode P. Wir können somit feststellen:

Satz 11. *Hat* (4) *genau ϱ linear unabhängige mit P periodische Lösungen, für deren jede* (6) *gilt, so besitzt Gleichung* (1) *eine ϱ-parametrige mit P periodische Lösungsschar.*

Dies Ergebnis steht in Einklang mit der trivialen Tatsache, daß man alle mit P periodischen Lösungsvektoren von (1) erhält, indem man zu einem von ihnen sämtliche mit P periodischen Lösungsvektoren von (3) hinzuaddiert.

Teil einer bei der Naturwissenschaft-Philosophischen Fakultät der Technischen Hochschule Braunschweig eingereichten Dissertation. Referenten: Professor Dr. R. Iglisch und Prof. Dr. H.-J. Kanold. Auch Herrn Dr. H. Eltermann bin ich für wertvolle Ratschläge zu Dank verpflichtet.

Mathematisches Institut
der Technischen Hochschule
Braunschweig

(Eingegangen am 15. September 1960)

Über den Resonanzfall bei Systemen von n nichtlinearen gewöhnlichen Differentialgleichungen erster Ordnung

Rahmi Ibrahim Ibrahim Abdel Karim

Vorgelegt von L. Collatz

§ 1. Problemstellung, Hauptergebnisse

In Erweiterung von Untersuchungen von R. Iglisch [*1*] wird das nichtlineare Differentialgleichungssystem

$$\frac{d u_i}{d t} = g_i(u_1, u_2, \ldots, u_n; t) + h_i(t) \qquad (i = 1, 2, \ldots, n) \tag{1}$$

betrachtet, das sich unter Einführung der Vektoren

$$\mathfrak{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad \mathfrak{h}(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ h_n(t) \end{pmatrix} \quad \text{und} \quad \mathfrak{g}(\mathfrak{u}, t) = \begin{pmatrix} g_1(u_1, \ldots, u_n, t) \\ g_2(u_1, \ldots, u_n, t) \\ \vdots \\ g_n(u_1, \ldots, u_n, t) \end{pmatrix} \tag{2}$$

in der Form

$$\frac{d\mathfrak{u}}{dt} = \mathfrak{g}(\mathfrak{u}, t) + \mathfrak{h}(t) \tag{3}$$

schreiben läßt. Es gelte die Periodizitätsvoraussetzung

$$\mathfrak{g}(\mathfrak{u}, t + P) = \mathfrak{g}(\mathfrak{u}, t), \qquad \mathfrak{h}(t + P) = \mathfrak{h}(t). \tag{4}$$

Für die Funktionen g_i sei hinsichtlich der Veränderlichen u_i eine Taylor-Entwicklung bis zu den Gliedern zweiter Ordnung möglich, $\mathfrak{h}(t)$ sei etwa stetig.

Es sei eine mit P periodische Lösung $\mathfrak{u}_0(t)$ von (3) bekannt:

$$\frac{d\mathfrak{u}_0}{dt} = \mathfrak{g}(\mathfrak{u}_0, t) + \mathfrak{h}(t), \qquad \mathfrak{u}_0(t + P) = \mathfrak{u}_0(t). \tag{5}$$

Untersucht werden dann bei geringer Abänderung von $\mathfrak{h}(t)$ die zu $\mathfrak{u}_0(t)$ benachbarten Lösungsvektoren $\mathfrak{u}(t)$ des Gleichungssystems

$$\frac{d\mathfrak{u}}{dt} = \mathfrak{g}(\mathfrak{u}, t) + \mathfrak{h}(t) + \beta\,\mathfrak{f}(t) \tag{6}$$

bei genügend kleinem $|\beta|$; dabei soll wieder gelten:

$$\mathfrak{f}(t + P) = \mathfrak{f}(t). \tag{7}$$

Mit

$$\mathfrak{u}(t) = \mathfrak{u}_0(t) + \mathfrak{x}(t) \tag{8}$$

geht (6) nach Subtraktion von (5) über in

$$\frac{d\mathfrak{x}}{dt} = \mathfrak{g}\left(\mathfrak{u}_0(t) + \mathfrak{x}(t), t\right) - \mathfrak{g}\left(\mathfrak{u}_0(t), t\right) + \beta\,\mathfrak{f}(t). \tag{9}$$

Untersucht werden hiervon die Lösungen mit kleinem $|\mathfrak{x}(t)|$ bei kleinem $|\beta|$.

Wie in §2, wo eine Umformung von (9) vorgenommen werden soll, gezeigt werden wird, spielt die Rolle des homogenen linearen Gleichungssystems[1] das System

$$\frac{d\mathfrak{y}}{dt} = \mathfrak{A}(t)\,\mathfrak{y}(t) \tag{10}$$

mit der Matrix $(i, k = 1, 2, \ldots, n)$

$$\mathfrak{A}(t) = \left(a_{ik}(t)\right) = \left(\frac{\partial g_i(\mathfrak{u}_0, t)}{\partial u_k}\right) = \left(\frac{\partial \mathfrak{g}(\mathfrak{u}_0, t)}{\partial u_1}, \frac{\partial \mathfrak{g}(\mathfrak{u}_0, t)}{\partial u_2}, \ldots, \frac{\partial \mathfrak{g}(\mathfrak{u}_0, t)}{\partial u_n}\right), \tag{11}$$

Offenbar ist

$$\mathfrak{A}(t + P) = \mathfrak{A}(t). \tag{12}$$

Das zu (10) adjungierte homogene System ist dann, wenn durch das hochgestellte T der Übergang zur transponierten Matrix gekennzeichnet wird:

$$\frac{d\mathfrak{z}}{dt} = -\mathfrak{A}^T(t)\,\mathfrak{z}(t). \tag{13}$$

Satz 1 *(Resonanzfall). Besitzt* (13) *einen mit P periodischen Lösungsvektor $\mathfrak{z}(t)$, für den*

$$\int_t^{t+P} \mathfrak{z}^T(t)\,\mathfrak{f}(t)\,dt = C \neq 0 \tag{14}$$

ausfällt, so nimmt jede Lösung $\mathfrak{x}(t)$ von (9) *— unabhängig von den Anfangswerten — mit wachsendem t Werte an, deren Absolutbeträge mindestens von der Größenordnung $\sqrt{|\beta|}$ sind. — Der Beweis wird in* §3 *erbracht.*

In §4 ergibt sich dann

Satz 2 *(Hauptfall). Hat* (13) *keinen mit P periodischen Lösungsvektor $\mathfrak{z}(t)$, so besitzt* (9) *zu jedem hinreichend kleinen β Lösungsvektoren $\mathfrak{x}(t)$, die für alle Werte von t von der Größenordnung $|\beta|$ bleiben, d.h. für die*

$$|\mathfrak{x}(t)| \leqq \text{Const.}\,|\beta| \tag{15}$$

gilt für alle t; z.B. die eindeutig existierende kleine mit P periodische Lösung:

$$\mathfrak{x}(t + P) = \mathfrak{x}(t).$$

Bleibt noch der Ausnahmefall übrig, daß zwar (13) mit P periodische Lösungsvektoren $\mathfrak{z}_1(t), \ldots, \mathfrak{z}_r(t)$ besitzt mit $1 \leqq r \leqq n$, daß aber für alle diese periodischen

[1] Vgl. hierzu die Arbeit [2] des Verfassers.

Lösungen

$$\int_t^{t+P} \mathfrak{z}_\varrho^T(t)\,\mathfrak{f}(t)\,dt = 0 \qquad (\varrho = 1, \ldots, r) \tag{16}$$

gilt. In diesem Fall lassen sich ohne weitere Voraussetzungen über die Funktionen $g_i(\mathfrak{u}, t)$ keine allgemeinen Aussagen machen.

§ 2. Eine Umformung

Setzt man

$$\mathfrak{u}(t, \lambda) = \mathfrak{u}_0(t) + \lambda\,\mathfrak{x}(t), \tag{17}$$

so ist

$$\mathfrak{u}(t, 0) = \mathfrak{u}_0(t), \qquad \mathfrak{u}(t, 1) = \mathfrak{u}_0(t) + \mathfrak{x}(t). \tag{18}$$

Wendet man auf die Differenz rechter Hand in der i-ten Komponentengleichung von (9) die Taylor-Entwicklung hinsichtlich λ an, so kommt

$$g_i(\mathfrak{u}_0 + \mathfrak{x}, t) = g_i(\mathfrak{u}_0, t) + \frac{d g_i(\mathfrak{u}_0, t)}{d\lambda} + \int_0^1 \frac{d^2 g_i(\mathfrak{u}(t, \lambda), t)}{d\lambda^2} \cdot (1 - \lambda)\,d\lambda$$

oder, wenn $x_k(t)$ die Komponenten des Vektors $\mathfrak{x}(t)$ sind,

$$\begin{aligned} g_i(\mathfrak{u}_0 + \mathfrak{x}, t) - g_i(\mathfrak{u}_0, t) &= \sum_{k=1}^{n} \frac{\partial g_i(\mathfrak{u}_0, t)}{\partial u_k}\,x_k + \\ &+ \int_0^1 \sum_{k,l}^{1\ldots n} \frac{\partial^2 g_i(\mathfrak{u}(t, \lambda), t)}{\partial u_k\,\partial u_l}\,x_k\,x_l\,(1 - \lambda)\,d\lambda. \end{aligned} \tag{19}$$

Wir führen nun neben der Matrix $\mathfrak{A}(t)$ aus (11) noch den „Tensor" $\mathsf{T}\big(\mathfrak{u}(t, \lambda), t\big)$ ein, dessen n Komponenten die Matrizen $(k, l = 1, \ldots, n)$

$$\mathfrak{T}_i\big(\mathfrak{u}(t, \lambda), t\big) = \left(\frac{\partial^2 g_i(\mathfrak{u}(t, \lambda), t)}{\partial u_k\,\partial u_l}\right), \qquad i = 1, \ldots, n \tag{20}$$

sein sollen:

$$\mathsf{T}\big(\mathfrak{u}(t, \lambda), t\big) = \begin{pmatrix} \mathfrak{T}_1\big(\mathfrak{u}(t, \lambda), t\big) \\ \mathfrak{T}_2\big(\mathfrak{u}(t, \lambda), t\big) \\ \vdots \\ \mathfrak{T}_n\big(\mathfrak{u}(t, \lambda), t\big) \end{pmatrix}. \tag{21}$$

Dann läßt sich (9) in der Form

$$\frac{d\mathfrak{x}(t)}{dt} = \mathfrak{A}(t)\,\mathfrak{x}(t) + \beta\,\mathfrak{f}(t) + \int_0^1 \mathfrak{x}^T(t)\,\mathsf{T}\big(\mathfrak{u}(t, \lambda), t\big)\,\mathfrak{x}(t)\,(1 - \lambda)\,d\lambda \tag{22}$$

schreiben. — Hiervon werden bei genügend kleinem ε und β_0 Lösungen gesucht mit

$$|\mathfrak{x}(t)| \leqq \varepsilon \quad \text{bei} \quad |\beta| \leqq \beta_0. \tag{23}$$

Es sei noch angemerkt, daß der Tensor

$$\mathsf{T}\big(\mathfrak{u}(t, 0), t\big) = \mathsf{T}\big(\mathfrak{u}_0(t), t\big),$$

die Periode P besitzt:

$$\mathsf{T}\big(\mathfrak{u}_0(t + P), t + P\big) = \mathsf{T}\big(\mathfrak{u}_0(t), t\big). \tag{24}$$

Aus dieser Tatsache in Verbindung mit der ersten Beziehung (23) folgt die Existenz einer endlichen Konstanten M derart, daß für das in (22) auftretende Integral

$$\left|\int_0^1 \mathfrak{x}^T(t)\,\mathsf{T}\left(\mathfrak{u}(t,\lambda),t\right)\mathfrak{x}(t)\,(1-\lambda)\,d\lambda\right| \leqq M\cdot\left(\max|\mathfrak{x}(t)|\right)^2 \tag{25}$$

gilt. Dazu braucht nur vorausgesetzt zu werden, daß alle rechter Hand in (20) auftretenden zweiten Ableitungen in einem Intervall $t_1 \leqq t \leqq t_1 + P$ für alle Werte $|\mathfrak{u}(t,\lambda)-\mathfrak{u}_0(t)| \leqq \varepsilon$ des ersten Arguments beschränkt sind.

§ 3. Der Resonanzfall

Das adjungierte System (13) besitze einen mit P periodischen Lösungsvektor $\mathfrak{z}(t)$, für den (14) gilt. Im Gegenteil zur Behauptung des Satzes 1 machen wir für eine beliebige Lösung $\mathfrak{x}(t)$, die der ersten Voraussetzung (23) genügt, zusätzlich noch die

$$\text{Annahme:}\quad |\mathfrak{x}(t)| \leqq N\sqrt{|\beta|} \tag{26}$$

für alle t mit einer später zu bestimmenden endlichen Konstanten N. Analog zur Formel (33) in [2] erhält man aus (22) und (13) für beliebiges positiv ganzes m mit C aus (14)

$$\begin{aligned} \mathfrak{z}^T(t)\left[\mathfrak{x}(t+mP)-\mathfrak{x}(t)\right] &= mC\cdot\beta + \\ &+\int_t^{t+mP}\mathfrak{z}^T(\tau)\int_0^1 \mathfrak{x}^T(\tau)\,\mathsf{T}\left(\mathfrak{u}(\tau,\lambda),\tau\right)\mathfrak{x}(\tau)\,(1-\lambda)\,d\lambda\,d\tau. \end{aligned} \tag{27}$$

Wegen der Periodizität von $\mathfrak{z}^T(t)$ gilt für alle t eine Beschränkung der Form

$$|\mathfrak{z}^T(t)| \leqq D. \tag{28}$$

Benutzt man (28), (26) und (25), so kann man nach (27) so abschätzen:

$$D\cdot 2N|\beta|^{\frac{1}{2}} + DMN^2|\beta|\,mP \geqq m|\beta|\,|C|$$

oder

$$2DN \geqq m|\beta|^{\frac{1}{2}}\cdot B \tag{29}$$

mit

$$B = |C| - DMN^2P > 0. \tag{30}$$

Setzt man für N eine hinreichend kleine Konstante ein, so ist B positiv. Nun kann man zu jedem $\beta \neq 0$ die Zahl m so groß wählen, daß (29) einen Widerspruch enthält. Damit ist die Annahme (26) in Verbindung mit der ersten Beziehung (23) als unmöglich erkannt, und es gilt der folgende

Satz 3. *Im Resonanzfall gibt es vier Konstanten ε, M, D, N, aus denen sich vermöge (30) ein positives B bestimmen läßt, derart, daß jede Lösung $\mathfrak{x}(t)$ von (9) in jedem Intervall der Länge $\frac{2\cdot DN}{B|\beta|^{\frac{1}{2}}}P$ mindestens einmal einen Wert annimmt, so daß*

$$|\mathfrak{x}(t)| > \operatorname{Min}\left(\varepsilon, N\sqrt{|\beta|}\right) \tag{31}$$

zu Recht besteht.

Zusatz. Beschränkt man β durch

$$|\beta| < \frac{\varepsilon^2}{N^2}, \quad \text{mit} \quad N^2 < \frac{|C|}{D\,M\,P} \qquad (\text{vgl. } (30)),$$

so gilt in jedem Intervall der Länge $\frac{2DN}{B|\beta|^{\frac{1}{2}}} P$ mindestens für einen Wert von t sogar die Abschätzung

(32) $$|\mathfrak{x}(t)| > N\sqrt{|\beta|}.$$

§ 4. Der Hauptfall

Wir benötigen im folgenden den Hauptsatz über implizite Funktionen als

Hilfssatz 1. Der Vektor

(33) $$\mathfrak{v}(\mathfrak{a},\beta) = \mathfrak{v}(a_1, a_2, \ldots, a_n, \beta) = \begin{pmatrix} v_1(a_1, \ldots, a_n, \beta) \\ v_2(a_1, \ldots, a_n, \beta) \\ \vdots \\ v_n(a_1, \ldots, a_n, \beta) \end{pmatrix}$$

besitze in einer gewissen Umgebung der Werte

$$a_1 = 0, \ a_2 = 0, \ \ldots, \ a_n = 0, \ \beta = 0$$

stetige erste Ableitungen nach den a_ν und sei in allen $n+1$ Veränderlichen stetig; außerdem sei

(34) $$\mathfrak{v}(0, \ldots, 0, 0) = \mathfrak{o};$$

schließlich sei die Funktionaldeterminante

(35) $$\left|\frac{\partial v_i(0, \ldots, 0, 0)}{\partial a_k}\right| \neq 0 \qquad (i, k = 1, 2, \ldots, n).$$

Dann gibt es zu jedem genügend kleinem a_0 eine Schranke β_0, derart, daß zu jedem β mit $|\beta| \leqq \beta_0$ eindeutig ein Lösungsvektor $\mathfrak{a} = \mathfrak{a}(\beta)$ des Gleichungssystems

(36) $$\mathfrak{v}(\mathfrak{a}, \beta) = \mathfrak{o}$$

existiert, für den $|\mathfrak{a}| \leqq a_0$ ist. Zum Beweis vgl. z.B. [3].

Die Anwendung dieses Hilfssatzes geschieht so: Man charakterisiere jeden Lösungsvektor $\mathfrak{x}(t)$ von (9) durch seine Anfangswerte:

(37) $$\mathfrak{x}(0) = \mathfrak{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

also

(38) $$\mathfrak{x} = \mathfrak{x}(\mathfrak{a}, \beta; t).$$

Dieser Vektor hat bezüglich t dann und nur dann die Periode P, wenn

(39) $$\mathfrak{v}(\mathfrak{a}, \beta) \equiv \mathfrak{x}(\mathfrak{a}, \beta; P) - \mathfrak{x}(\mathfrak{a}, \beta; 0) = \mathfrak{o}$$

ist; dies entspricht der Behauptung (36) des Hilfssatzes. Für $\beta=0$ ist der Nullvektor Lösung von (9); also ist, da dann auch $\mathfrak{a}$ der Nullvektor ist, (34) erfüllt. Die Ableitungs- und Stetigkeitsvoraussetzungen des Hilfssatzes sind nach bekannten Sätzen über die Abhängigkeit der Lösungsvektoren von (9) von den Anfangswerten $\mathfrak{a}$ und dem Parameter β gleichfalls gesichert (vgl. z.B. [4]) durch folgenden

Hilfssatz 2. Im Intervall $0 \leqq t \leqq P$ kann man zu vorgegebenem ε je ein positives ε_1 und β_1 bestimmen derart, daß die Lösungen $\mathfrak{x}(t)$ von (9) im ganzen Intervall der Abschätzung $|\mathfrak{x}(t)| \leqq \varepsilon$ genügen, sofern nur $|\mathfrak{x}(0)| \leqq \varepsilon_1$ und $|\beta| \leqq \beta_1$ gewählt wird.

Gemäß (39) muß zu Anwendung des Hilfssatzes 1 noch dafür gesorgt werden, daß sowohl $\varepsilon_1 \leqq a_0$ als auch $\beta_1 \leqq \beta_0$ ist. — Differentiation von (9) nach a_l liefert, daß die Vektoren

$$\mathfrak{y}_l^*(t) = \frac{\partial \mathfrak{x}(\mathfrak{a}, \beta; t)}{\partial a_l} \qquad (l = 1, 2, \ldots, n) \tag{40}$$

Lösungen des Gleichungssystems

$$\frac{d\mathfrak{y}^*}{dt} = \mathfrak{B}(t)\,\mathfrak{y}^* \tag{41}$$

sind mit der Matrix

$$\mathfrak{B}(t) = \left(\frac{\partial g_i(\mathfrak{u}_0(t) + \mathfrak{x}(t), t)}{\partial u_k}\right) \qquad (i, k = 1, \ldots, n). \tag{42}$$

Daher sind die Vektoren

$$\mathfrak{y}_l^0(t) = \frac{\partial \mathfrak{x}(0, \ldots, 0, 0; t)}{\partial a_l} \qquad (l = 1, \ldots, n) \tag{43}$$

Lösungen von (10), da dann (42) in (11) übergeht. Die Vektoren (43) sind linear unabhängig, weil ihre Determinante für $t=0$ die Einheitsdeterminante ist (vgl. hierzu [2], Satz 4). Da wir hier im Hauptfall die Voraussetzung machen, daß (13) und damit auch (10) (vgl. [2], Satz 8) keine mit P periodischen Lösungsvektoren besitzt, so ist (vgl. [2], Satz 10)

$$\operatorname{Det}\begin{pmatrix} \mathfrak{y}_1^0(\tau)\big|_t^{t+P} \\ \mathfrak{y}_2^0(\tau)\big|_t^{t+P} \\ \vdots \\ \mathfrak{y}_n^0(\tau)\big|_t^{t+P} \end{pmatrix} \neq 0. \tag{44}$$

Ein Blick auf (39) lehrt, daß dies gerade die Bedingung (35) des Hilfssatzes ist. Da alle Bedingungen des Hilfssatzes als gültig nachgewiesen sind, ist seine Behauptung, bei uns also (39), bewiesen.

Es gilt daher

Satz 4. *Hat (13) keinen mit P periodischen Lösungsvektor $\mathfrak{z}(t)$, so gibt es zu vorgegebenem ε Schranken ε_1 und β_1 derart, daß zu jedem β mit $|\beta| \leqq \beta_1$ ein Anfangsvektor $\mathfrak{x}(0)$ mit $|\mathfrak{x}(0)| \leqq \varepsilon_1$ existiert, so daß die Lösung $\mathfrak{x}(t)$ mit diesem Anfangswert $\mathfrak{x}(0)$ periodisch ist und der gleichmäßigen Abschätzung $|\mathfrak{x}(t)| \leqq \varepsilon$ genügt.*

Dies ist ein Teil des Satzes 2. Um Satz 2 vollständig zu beweisen, benötigen wir noch den

Satz 5. *Es existiert eine endliche Konstante E derart, daß bei genügend kleinem* $|\beta|$ *für diese mit P periodische Lösung* $\mathfrak{x}(t)$ *von* (9) *gilt*

$$|\mathfrak{x}(t)| \leqq E \cdot |\beta|. \tag{45}$$

Beweis. Sind $\mathfrak{z}_1(t), \ldots, \mathfrak{z}_n(t)$ linear unabhängige Lösungen von (13), so erhält man unter Ausnutzung der Periodizität von $\mathfrak{x}(t)$ analog der Gleichung [2], (38) für $\nu = 1, \ldots, n$

$$\begin{aligned}[\mathfrak{z}_\nu^T(t+P) - \mathfrak{z}_\nu^T(t)]\,\mathfrak{x}(t) &= \beta \int_t^{t+P} \mathfrak{z}_\nu^T(\tau)\,\mathfrak{f}(\tau)\,d\tau + \\ &+ \int_t^{t+P} \mathfrak{z}_\nu^T(\tau) \int_0^1 \mathfrak{x}^T(\tau)\,\mathsf{T}\big(\mathfrak{u}(\tau,\lambda),\tau\big)\,\mathfrak{x}(\tau)\,(1-\lambda)\,d\lambda\,d\tau\end{aligned} \tag{46}$$

(vgl. dazu auch Gl. (27)). Die Determinante der Koeffizientenmatrix linker Hand ist nach der Voraussetzung des Hauptfalles von Null verschieden (vgl. (44)). Daher kann man das lineare Gleichungssystem (46) für die linker Hand stehenden Komponenten $x_1(t), \ldots, x_n(t)$ von $\mathfrak{x}(t)$ nach der Cramerschen Auflösungsformel auflösen. Bezeichnet bei festgehaltenem t

$$x = \operatorname*{Max}_{t \leqq \tau \leqq t+P} |x_\nu(\tau)| \qquad (\nu = 1, 2, \ldots, n), \tag{47}$$

so findet man auf diese Weise eine Abschätzung der Form

$$x \leqq K|\beta| + L \cdot \big(\max|\mathfrak{x}(t)|\big)^2 \tag{48}$$

mit zwei endlichen Konstanten K und L, wobei (25) beachtet wurde.

Da

$$\max|\mathfrak{x}(t)| \leqq \sqrt{n}\,x$$

ist, folgt aus (48) erst recht

$$\max|\mathfrak{x}(t)| \leqq \sqrt{n}\,K|\beta| + \sqrt{n}\,L \cdot \big(\max|\mathfrak{x}(t)|\big)^2,$$

also

$$\max|\mathfrak{x}(t)|\,\big[1 - \sqrt{n}\,L \cdot (\max|\mathfrak{x}(t)|)\big] \leqq \sqrt{n}\,K|\beta|.$$

Auf Grund von Satz 4 dürfen wir voraussetzen, daß die eckige Klammer positiv ist. Für $\varepsilon \leqq \frac{1}{2\sqrt{n}\,L}$ z.B. wird sie $\geqq \frac{1}{2}$. Daraus ergibt sich sofort die Abschätzung (45) mit $E = 2\sqrt{n}\,K$.

Statt die Cramersche Auflösungsformel anzuwenden, kann man natürlich auch (46) durch linksseitige Multiplikation mit der zu $[\mathfrak{z}_\nu^T(t+P) - \mathfrak{z}_\nu^T(t)]$ inversen Matrix nach $\mathfrak{x}(t)$ auflösen und dann mit Hilfe der Schwarzschen Ungleichung $|\mathfrak{x}(t)|$ abschätzen. Auch so gelangt man zu (45).

II. Teil einer bei der Naturwissenschaftlich-Philosophischen Fakultät der Technischen Hochschule zu Braunschweig eingereichten Dissertation. Referenten: Prof. Dr. R. Iglisch und Prof. Dr. H.-J. Kanold. Auch Herrn Dr. H. Eltermann bin ich für wertvolle Ratschläge zu großem Dank verpflichtet.

Literatur

[1] Iglisch, R.: Der Resonanzfall bei nichtlinearen gewöhnlichen Differentialgleichungen zweiter Ordnung. Arch. Rational Mech. Anal. **3**, 187–193 (1959).
[2] Karim, R. I. A.: Über den Resonanzbegriff bei Systemen von n linearen gewöhnlichen Differentialgleichungen erster Ordnung. Arch. Rational Mech. Anal. **6**, 21–28 (1961).
[3] v. Mangoldt-Knopp: Einführung in die Höhere Mathematik, 10. Aufl., Bd. II, Nr. 134. Stuttgart: Hirzel 1956.
[4] Kamke, E.: Differentialgleichungen reeller Funktionen, 3. Aufl., § 17. Leipzig: Akademische Verlagsgesellschaft 1956.

Mathematisches Institut
der Technischen Hochschule
Braunschweig

(Eingegangen am 15. September 1960)

A Runge-Kutta Procedure for the Goursat Problem in Hyperbolic Partial Differential Equations

ROBERT H. MOORE

Communicated by L. CESARI

Contents

		Page
§1.	Introduction	37
§2.	Taylor Expansions	39
§3.	Definitions of the Matrix Quantities to be Used	40
§4.	The Basic Equations for the Runge-Kutta Procedure for 2-HP	43
§5.	Computations Preparatory to Matching the Taylor Expansions	44
§6.	Requirements on the Parameters in the Case for u	48
§7.	Requirements on the Parameters in the Case for p	51
§8.	Requirements on the Parameters in the Case for q	53
§9.	Further Requirements	54
§10.	Values for the Parameters	55
§11.	Determination of $\boldsymbol{U}_0$ and $\boldsymbol{B}_\lambda^\omega$	57
§12.	Convergence of the Numerical Approximations Obtained by R-K 2	59

§1. Introduction

The subject of this paper[1] is the development of a numerical procedure for finding an approximate solution of the Goursat Problem:

$$u_{xy} = f(x, y, u, u_x, u_y), \tag{1.1}$$

$$u(x, 0) = \sigma(x), \quad u(0, y) = \tau(y), \quad \sigma(0) = \tau(0), \tag{1.2}$$

$$0 \leqq x \leqq a, \quad 0 \leqq y \leqq b. \tag{1.3}$$

We shall denote this problem by 2-HP.

The usual Runge-Kutta procedure, which we denote by R-K 1, applies to the problem of finding a function $y(x)$ satisfying

$$\frac{dy}{dx} = f(x, y), \quad y(0) = y_0; \quad 0 \leqq x \leqq a. \tag{1.4}$$

The procedure developed here, denoted by R-K 2, is analogous to R-K 1 both in mode of application and order of accuracy with respect to Taylor series expansions. Just as the usual Runge-Kutta procedure is a refinement of the

[1] The research presented here was supported by the Office of Scientific Research, Air Research and Development Command, under contract No. 49(638)-192 and by the United States Naval Ordnance Laboratory.

Euler-Cauchy polygon method for (1.4), so also R-K 2 is a refinement of DIAZ's [1] analogue for 2-HP of the Euler-Cauchy polygon method. Again, just as the validity of R-K 1, in the sense of uniform convergence to a true solution, depends on the Peano existence theorem for (1.4), so also the validity of R-K 2 depends on an analogue for 2-HP of PEANO's theorem, given in [3].

Concerning regularity, it will be assumed that f is three times continuously differentiable with respect to any combination of its arguments, that f and its first derivatives are bounded, and that σ and τ are five times continuously differentiable throughout any domains in which they are used. Naturally, if a lower order of accuracy than that given here is acceptable, then these regularity assumptions may be relaxed. While these assumptions on σ and τ are necessary in order to assure the stated accuracy, only tabulated values of σ, σ', τ, and τ' at suitable points are needed in order to carry out the actual numerical computations.

It is suggestive to consider the system of integral equations equivalent to (1.1) and (1.2):

$$\begin{aligned} u(x,y) &= \sigma(x)+\tau(y)-\sigma(0)+\int_0^x\int_0^y f(\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta))\,d\xi\,d\eta, \\ p(x,y) &= \sigma'(x)+\int_0^y f(x,\eta,u(x,\eta),p(x,\eta),q(x,\eta))\,d\eta, \\ q(x,y) &= \tau'(y)+\int_0^x f(\xi,y,u(\xi,y),p(\xi,y),q(\xi,y))\,d\xi. \end{aligned} \tag{1.5}$$

The procedure to be described can be viewed as a technique of stepwise approximate integration of these equations.

Thus, the overall numerical solution of the problem in a rectangle R: $0\leq x\leq a$, $0\leq y\leq b$, is to be carried out in a stepwise manner over a rectangular mesh on R. The object of R-K 2 is to provide a method for obtaining approximate values of u, u_x and u_y at $(x_0+h,\ y_0+\varphi h)$ given the values at (x_0, y_0), (x_0+h, y_0) and $(x_0, y_0+\varphi h)$ where these points are the corners of a rectangle of the mesh. In terms of the Taylor expansions of u, $p=u_x$ and $q=u_y$ about (x_0, y_0), these values are to be accurate through terms of order h^4, h^3 and h^3, respectively. This implies tacitly that φ is bounded.

The solution of the differential equation (1.1) on the mesh subrectangle determined by (x_0, y_0) and $(x_0+h,\ y_0+\varphi h)$ is a "miniature" problem of type 2-HP. For the derivation of R-K 2, one assumes that the values of u, u_x and u_y at (x_0, y_0), (x_0+h, y_0) and $(x_0, y_0+\varphi h)$, as given from computations on the preceding subrectangles, are exactly the true values for the solution of 2-HP on R. The functions $s(x)$ and $t(x)$ defined by

$$s(x)=u(x,y_0), \qquad t(y)=u(x_0,y) \tag{1.6}$$

serve as the initial conditions for the miniature problem; they have the same differentiability properties as the true solution u, so $u_x=p=s'$ and $u_y=q=t'$. The original initial conditions given by σ and τ are involved only implicitly in the formulas. However, for subrectangles along the initial lines (those adjacent to $x=0$ and $y=0$), explicit use will have to be made of σ and τ (*cf.* eqns. (3.8), and particularly §11).

The basic idea of the derivation is then the same as that for R-K 1: An explicit computational procedure is outlined in which there are several free parameters; here these are the elements of certain diagonal matrices (*cf.* equations (3.13)) as well as quantities permitting consideration of only part of the subrectangle with sides h and φh (*cf.* §3). The values of u, p and q at $(x_0+h, y_0+\varphi h)$ are expressed as power series in h in terms of these parameters. Then, by matching the coefficients of the lower order powers of h in this expression and in the Taylor expansions of u, p and q about (x_0, y_0), a system of equations is derived with the parameters as unknowns (*cf.* equations (R 1)—(R 11) in §§ 6—8). A solution of the system, used in the computational procedure, will then yield values for u, p and q at $(x_0+h, y_0+\varphi h)$ of a prescribed order of accuracy.

§ 2. Taylor Expansions

Under the assumed regularity of f, σ and τ there is a unique solution $u(x, y)$ of 2-HP. With this solution, let

$$F(x, y) = f\big(x, y, u(x, y), u_x(x, y), u_y(x, y)\big).$$

Then F_x, F_y, F_{xx}, *etc.* denote the partial derivatives of F. Note that, from the differential equation, $F_x = u_{xxy}$, for example.

If one writes the Taylor expansion about (x_0, y_0) for $u(x_0+h, y_0+\varphi h)$ and makes use of the last remark and the definitions of s, t and F, then one finds

$$\begin{aligned}
\Delta u &= u(x_0+h, y_0+\varphi h) - u(x_0, y_0)\\
&= h s' + \frac{h^2}{2} s'' + \frac{h^3}{3!} s''' + \frac{h^4}{4!} s^{(4)} +\\
&\quad + \varphi h t' + \frac{\varphi^2 h^2}{2} t'' + \frac{\varphi^3 h^3}{3!} t''' + \frac{\varphi^4 h^4}{4!} t^{(4)} + \\
&\quad + \varphi h^2 f + \varphi h^3 \left[\frac{1}{2} F_x + \frac{1}{2} \varphi F_y\right] + \\
&\quad + \varphi h^4 \left[\frac{1}{6} F_{xx} + \frac{1}{4} \varphi F_{xy} + \frac{1}{6} \varphi^2 F_{yy}\right] + \mathcal{O}(h^5).
\end{aligned} \tag{2.1}$$

In this, the various coefficients of the powers of h are to be evaluated at (x_0, y_0); in particular, the f in $\varphi h^2 f$ means $F(x_0, y_0)$.

Similarly, for the function $p = u_x$, since $p_y(x, y) = F(x, y)$ and $p(x, y_0) = s'(x)$,

$$\begin{aligned}
\Delta p &= p(x_0+h, y_0+\varphi h) - p(x_0, y_0)\\
&= h s'' + \frac{h^2}{2} s''' + \frac{h^3}{3!} s^{(4)} + \varphi h f + \varphi h^2 \left[F_x + \frac{1}{2} \varphi F_y\right] + \\
&\quad + \varphi h^3 \left[\frac{1}{2} F_{xx} + \frac{1}{2} \varphi F_{xy} + \frac{1}{6} \varphi^2 F_{yv}\right] + \mathcal{O}(h^4).
\end{aligned} \tag{2.2}$$

In the same manner, one finds

$$\begin{aligned}
\Delta q &= \varphi h t'' + \frac{(\varphi h)^2}{2} t''' + \frac{(\varphi h)^3}{3!} t^{(4)} + h f + h^2 \left[\frac{1}{2} F_x + \varphi F_y\right] + \\
&\quad + h^3 \left[\frac{1}{6} F_{xx} + \frac{1}{2} \varphi F_{xy} + \frac{1}{2} \varphi^2 F_{yy}\right] + \mathcal{O}(h^4).
\end{aligned} \tag{2.3}$$

§ 3. Definitions of the Matrix Quantities to be Used

As in the one dimensional case, it is not very practical for numerical work to use these Taylor expansions directly to determine the increments Δu, Δp and Δq. Rather, the various terms will be matched by the procedure below to within the desired order of accuracy. For this purpose, a matrix notation will be employed.

Let

$$(3.1)\qquad \boldsymbol{U}=\begin{pmatrix} x\\ y\\ u\\ p\\ q\end{pmatrix},\qquad \boldsymbol{\Phi}(\boldsymbol{U})=\begin{pmatrix}1\\ 1\\ f(\boldsymbol{U})\\ f(\boldsymbol{U})\\ f(\boldsymbol{U})\end{pmatrix},\qquad \mathscr{D}=\begin{pmatrix}\frac{\partial}{\partial x} & 0 & 0 & 0 & 0\\ 0 & \frac{\partial}{\partial y} & 0 & 0 & 0\\ 0 & 0 & \frac{\partial^2}{\partial x\,\partial y} & 0 & 0\\ 0 & 0 & 0 & \frac{\partial}{\partial y} & 0\\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial x}\end{pmatrix}$$

in which $f(\boldsymbol{U})$ has the value $f(x, y, u, p, q)$. Then the differential equation (1.1) can be written

$$(3.2)\qquad \mathscr{D}\,\boldsymbol{U}=\boldsymbol{\Phi}(\boldsymbol{U}).$$

It is to be noted that in general the first two components of vectors and diagonal matrices are thus associated with the independent variables x and y, while the latter three components are associated with the dependent variables u, p and q, respectively.

The initial values at (x_0, y_0) are expressed by the "initial vector"

$$(3.3)\qquad \boldsymbol{U}_0=\begin{pmatrix}x_0\\ y_0\\ u_0\\ p_0\\ q_0\end{pmatrix}.$$

The initial conditions along the boundaries $x=x_0$ and $y=y_0$ of the subrectangle under consideration do not need to be given in continuous form; indeed, because the stepwise numerical process gives only values at the mesh nodes, only finite differences are known. The vector expressing the conditions along $x=x_0$ and $y=y_0$ is thus taken to be

$$(3.4)\qquad \boldsymbol{B}=\begin{pmatrix}0\\ 0\\ \Delta s+\Delta t\\ \Delta s'\\ \Delta t'\end{pmatrix}$$

where

$$\Delta s=s(x_0+h)-s(x_0),\qquad \Delta t=t(y_0+\varphi h)-t(y_0),$$
$$\Delta s'=s'(x_0+h)-s'(x_0),\qquad \Delta t'=t'(y_0+\varphi h)-t'(y_0).$$

There are three things which should be noted here which point out the modifications of R-K 1 needed for 2-HP, and in particular make plausible the choice of the increment vectors $\boldsymbol{H}_\lambda^\omega$ below. First, as suggested by the fact that there are three unknown functions, *viz* u, p and q, three cases will be distinguished, one for each of u, p and q. That is, it will in general be necessary to match separately the Taylor series expansions of Δu, Δp and Δq. (However, see the comment at the end of § 9 concerning the possibility of other solutions (not now known) for parameters satisfying the requirements (R 1 ω)—(R 11 ω) of §§ 6, 7, 8.) Matrices and their elements used for a particular case will be distinguished by a superscript u, p or q; the superscript ω will stand for any one of u, p and q. For notational convenience, the superscript will often be suppressed if no confusion can result. In particular, if a matrix has a superscript ω, then each of its elements is assumed also to be so indexed, even though the ω is suppressed on the elements.

Second, because in the Taylor expansions for Δu, Δp and Δq the terms involving second derivatives of F are three in number, enough freedom is needed to match all three terms. (In the Taylor expansion of $y(x)$ used in the derivation of R-K 1 for $y' = f(x, y(x))$, there is only one term involving the second derivative of $F = f(x, y(x))$.) There will be three equations for matching these coefficients, *viz* (R 4 ω), (R 5 ω), (R 6 ω) of §§ 6—8. For this reason, entities will be introduced with a subscript λ where $\lambda = 1, 2, 3$. The greater degree of freedom thus afforded will permit satisfying the mentioned requirements.

Third, if R-K 1 is regarded as a procedure for approximate integration of the integral equation equivalent to the differential equation (1.4), then the increment coefficient h used in R-K 1 is seen to be a measure of one subinterval of the domain of integration. From the integral equations (1.5) it is clear then that the corresponding increment coefficients to be used for u, p and q in R-K 2 are φh^2, φh and h, respectively. These appear as the last three components of $\boldsymbol{H}_\lambda^\omega$ below.

Thus increment matrices $\boldsymbol{H}_\lambda^\omega$ are defined by first setting

$$\varphi_\lambda^\omega = e_\lambda^\omega \varphi \quad \text{and} \quad h_\lambda^\omega = g_\lambda^\omega h \qquad (\omega = u, p, q;\ \lambda = 1, 2, 3) \tag{3.5}$$

where e_λ^ω and g_λ^ω are parameters to be determined. Then one defines

$$\boldsymbol{H}_\lambda^\omega = \operatorname{diag}\begin{pmatrix} h_\lambda \\ \varphi_\lambda h_\lambda \\ \varphi_\lambda h_\lambda^2 \\ \varphi_\lambda h_\lambda \\ h_\lambda \end{pmatrix}. \tag{3.6}$$

(Note that the ω has been suppressed.)

This definition of the $\boldsymbol{H}_\lambda^\omega$ then implies integration over only a *portion* of the subrectangle S determined by (x_0, y_0) and $(x_0 + h, y_0 + \varphi h)$; that is, subrectangles S_λ^ω determined by (x_0, y_0) and $(x_0 + h_\lambda^\omega, y_0 + \varphi_\lambda^\omega h_\lambda^\omega)$ are also considered (*cf.* § 9). By requiring

$$0 < g_\lambda^\omega \leqq 1 \quad \text{and} \quad 0 < e_\lambda^\omega g_\lambda^\omega \leqq 1, \tag{3.7}$$

one is assured that each S_λ^ω is contained in S (and is nondegenerate). For these smaller subrectangles S_λ^ω, further initial vectors corresponding to $\boldsymbol{B}$ will be needed. They are, to begin with, considered as a simple "cutting down" of $\boldsymbol{B}$ to S_λ^ω by means of linear interpolation. Thus $s(x_0+h_\lambda^\omega)-s(x_0)$, which might be thought of as $\Delta_\lambda^\omega s$, is replaced by $g_\lambda^\omega \Delta s$. By the mean value theorem of differential calculus, this can then be replaced by $g_\lambda^\omega \cdot h\bar{s}'$, that is, by $h_\lambda^\omega \bar{s}'$, where $\bar{s}'$ is a suitable "intermediate value" of $s'(x)$. It turns out, in the derivation, to be necessary to define also certain other intermediate values $\tilde{s}', \tilde{s}'', \tilde{t}', \tilde{t}''$. (See the comments following (5.18) and at the end of § 6.) The definitions for all of these are given by:

$$
\begin{aligned}
&h\bar{s}' = s(x_0+h)-s(x_0), && \tfrac{2}{3}h\tilde{s}' = s(x_0+\tfrac{2}{3}h)-s(x_0),\\
&h\bar{s}'' = s'(x_0+h)-s'(x_0), && \tfrac{2}{3}h\tilde{s}'' = s'(x_0+\tfrac{2}{3}h)-s'(x_0),\\
&\varphi h\bar{t}' = t(y_0+\varphi h)-t(y_0), && \tfrac{2}{3}\varphi h\tilde{t}' = t(y_0+\tfrac{2}{3}\varphi h)-t(y_0),\\
&\varphi h\bar{t}'' = t'(y_0+\varphi h)-t'(y_0), && \tfrac{2}{3}\varphi h\tilde{t}'' = t'(y_0+\tfrac{2}{3}\varphi h)-t'(y_0).
\end{aligned}
\tag{3.8}
$$

The numerical determination of $\tilde{s}', \tilde{s}'', \tilde{t}'$ and $\tilde{t}''$ to within the required order of accuracy so as to be able to prove the convergence of the approximate solutions is one of the more delicate parts of R-K 2. It will be discussed in detail in § 11.

Using the above, one defines the $\boldsymbol{B}_\lambda^\omega$ as follows:

$$
\boldsymbol{B}_\lambda^u = \begin{pmatrix} 0 \\ 0 \\ h_\lambda^u \tilde{s}' + \varphi_\lambda^u h_\lambda^u \tilde{t}' \\ h_\lambda^u \tilde{s}'' \\ \varphi_\lambda^u h_\lambda^u \tilde{t}'' \end{pmatrix},
\tag{3.9}
$$

$$
\boldsymbol{B}_\lambda^p = \begin{pmatrix} 0 \\ 0 \\ g_\lambda^p \Delta s + \varphi_\lambda^p h_\lambda^p \tilde{t}' \\ g_\lambda^p \Delta s' \\ \varphi_\lambda^p h_\lambda^p \tilde{t}'' \end{pmatrix}, \qquad
\boldsymbol{B}_\lambda^q = \begin{pmatrix} 0 \\ 0 \\ h_\lambda^q \tilde{s}' + e_\lambda^q g_\lambda^q \Delta t \\ h_\lambda^q \tilde{s}'' \\ e_\lambda^q g_\lambda^q \Delta t' \end{pmatrix}.
$$

These are the best forms of the definition for numerical purposes, but for the derivation to follow it is more convenient to replace Δs, Δt, $\Delta s'$ and $\Delta t'$ via (3.8) so as to write these simultaneously:

$$
\boldsymbol{B}_\lambda^\omega = \begin{pmatrix} 0 \\ 0 \\ h_\lambda^\omega \hat{s}' + \varphi_\lambda^\omega h_\lambda^\omega \hat{t}' \\ h_\lambda^\omega \hat{s}'' \\ \varphi_\lambda^\omega h_\lambda^\omega \hat{t}'' \end{pmatrix}
\tag{3.10}
$$

in which $\hat{}$ means $-$ or $\sim$ as given by (3.9) according to which case, u, p or q, is under consideration.

Differences such as $\bar{s}' - s'(x_0)$ will be important in the sequel. For brevity, let s', s'', t', t'' stand for $s'(x_0)$, $s''(x_0)$, $t'(y_0)$, $t''(y_0)$, respectively. Then, by two

term Taylor expansions, it follows from (3.8) that

$$
(3.11)\quad
\begin{aligned}
\bar{s}' - s' &= \vartheta_1 h s'' + \mathcal{O}(h^2), & \tilde{s}' - s' &= \tilde{\vartheta}_1 \tfrac{2}{3} h s'' + \mathcal{O}(h^2),\\
\bar{s}'' - s'' &= \vartheta_2 h s''' + \mathcal{O}(h^2), & \tilde{s}'' - s'' &= \tilde{\vartheta}_2 \tfrac{2}{3} h s''' + \mathcal{O}(h^2),\\
\bar{t}' - t' &= \vartheta_3 \varphi h t'' + \mathcal{O}(h^2), & \tilde{t}' - t' &= \tilde{\vartheta}_3 \tfrac{2}{3} \varphi h t'' + \mathcal{O}(h^2),\\
\bar{t}'' - t'' &= \vartheta_4 \varphi h t''' + \mathcal{O}(h^2), & \tilde{t}'' - t'' &= \tilde{\vartheta}_4 \tfrac{2}{3} \varphi h t''' + \mathcal{O}(h^2),
\end{aligned}
$$

where $0<\vartheta_i,\ \tilde{\vartheta}_i<1$. This last simple statement about ϑ_i and $\tilde{\vartheta}_i$ is not specific enough for later purposes. It will, however, be enough to use the known fact (see, *e.g.* PRASAD [*4*])

$$
(3.12)\qquad \vartheta_i = \tfrac{1}{2} + \mathcal{O}(h) \quad \text{and} \quad \tilde{\vartheta}_i = \tfrac{1}{2} + \mathcal{O}(h).
$$

Besides the parameters e_λ^ω and q_λ^ω so far introduced, further free parameters are introduced by the matrices (defined for $\omega = u, p, q;\ \lambda = 1, 2, 3$):

$$
(3.13)\quad
\begin{aligned}
\boldsymbol{V}_\lambda^\omega &= \operatorname{diag}(V_{\lambda.1}, V_{\lambda.2}, V_{\lambda.3}, V_{\lambda.4}, V_{\lambda.5}),\\
\boldsymbol{W}_{\lambda i}^\omega &= \operatorname{diag}(W_{\lambda i.1}, W_{\lambda i.2}, W_{\lambda i.3}, W_{\lambda i.4}, W_{\lambda i.5}) \qquad (i = 1, 2),\\
\boldsymbol{A}_{\lambda i}^\omega &= \operatorname{diag}(A_{\lambda i.1}, A_{\lambda i.2}, A_{\lambda i.3}, A_{\lambda i.4}, A_{\lambda i.5}) \qquad (i = 1, 2, 3).
\end{aligned}
$$

Here, $T_{i.j}$ means the j^{th} component of the vector or diagonal matrix $\boldsymbol{T}_i$.

§ 4. The Basic Equations for the Runge-Kutta Procedure for 2-HP

As mentioned, three cases are to be distinguished, one for each of u, p and q, in the following procedure. In each case three sets of computations are to be carried out, one for each of $\lambda = 1$, 2 and 3 and used in equation (4.3). (However, in the actual numerical work, the particular system of parameters found in § 10 permits one set of computations for each of the cases for p and q to be eliminated. In the following derivation this will be ignored.)

The equations below setting forth the computational scheme are very similar to those for R-K 1; indeed they are almost identical if the equations for R-K 1 are written in the analogous matrix notation. Besides the indices ω and λ, the principal alteration is the introduction of the $\boldsymbol{B}$ and $\boldsymbol{B}_\lambda^\omega$ matrices which are not present for R-K 1. The equations are

$$
(4.1)\quad
\begin{aligned}
\boldsymbol{K}_{\lambda 1}^\omega &= \boldsymbol{H}_\lambda^\omega \boldsymbol{\Phi}(\boldsymbol{U}_0), & \boldsymbol{U}_{\lambda 1}^\omega &= \boldsymbol{U}_0 + \boldsymbol{V}_\lambda^\omega (\boldsymbol{K}_{\lambda 1}^\omega + \boldsymbol{B}_\lambda^\omega),\\
\boldsymbol{K}_{\lambda 2}^\omega &= \boldsymbol{H}_\lambda^\omega \boldsymbol{\Phi}(\boldsymbol{U}_{\lambda 1}^\omega), & \boldsymbol{U}_{\lambda 2}^\omega &= \boldsymbol{U}_0 + \sum_{i=1}^{2} \boldsymbol{W}_{\lambda i}^\omega (\boldsymbol{K}_{\lambda i}^\omega + \boldsymbol{B}_\lambda^\omega),\\
\boldsymbol{K}_{\lambda 3}^\omega &= \boldsymbol{H}_\lambda^\omega \boldsymbol{\Phi}(\boldsymbol{U}_{\lambda 2}^\omega).
\end{aligned}
$$

To define the weighted average corresponding to that in R-K 1, first let

$$
(4.2)\qquad \boldsymbol{K}_{\lambda i}^* = \begin{pmatrix} h \\ \varphi h \\ K_{\lambda i.3}^u \\ K_{\lambda i.4}^p \\ K_{\lambda i.5}^q \end{pmatrix},
$$

and then define

$$\boldsymbol{\Delta} = \left(\sum_{\substack{\lambda=1,2,3 \\ i=1,2,3}} \boldsymbol{A}_{\lambda i} \boldsymbol{K}^*_{\lambda i} \right) + \boldsymbol{B}. \tag{4.3}$$

The approximate increments in u, p and q are then taken to be the last three components of $\boldsymbol{\Delta}$, respectively:

$$\begin{aligned} u(x_0 + h, y_0 + \varphi h) &= u(x_0, y_0) + \Delta_{.3}, \\ p(x_0 + h, y_0 + \varphi h) &= p(x_0, y_0) + \Delta_{.4}, \\ q(x_0 + h, y_0 + \varphi h) &= q(x_0, y_0) + \Delta_{.5}. \end{aligned} \tag{4.4}$$

§ 5. Computations Preparatory to Matching the Taylor Expansions

It is the purpose of this section to derive detailed expressions for the elements in each $\boldsymbol{K}^\omega_{\lambda i}$, and, further, to examine the differential operators which are employed.

The superscript ω will be suppressed throughout the section.

To begin with, recall the general Taylor expansion for a function of five independent variables: If the differential operator D is defined by

$$D = \sum_{i=1}^{5} r_i \frac{\partial}{\partial \alpha_i},$$

then[2]

$$f(\alpha_1 + r_1, \alpha_2 + r_2, \alpha_3 + r_3, \alpha_4 + r_4, \alpha_5 + r_5) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5). \tag{5.1}$$

It should be noted that the values of f, F, s, t and their derivatives in the following equations are those at (x_0, y_0) unless otherwise indicated.

From the definitions in §§ 3 and 4, one has

$$\boldsymbol{K}_{\lambda 1} = \boldsymbol{H}_\lambda \boldsymbol{\Phi}(\boldsymbol{U}_0) = \begin{pmatrix} h_\lambda \\ \varphi_\lambda h_\lambda \\ \varphi_\lambda h^2_\lambda f \\ \varphi_\lambda h_\lambda f \\ h_\lambda f \end{pmatrix} \tag{5.2}$$

and

$$\boldsymbol{U}_{\lambda 1} = \boldsymbol{U}_0 + \boldsymbol{V}_\lambda (\boldsymbol{K}_{\lambda 1} + \boldsymbol{B}_\lambda) = \begin{pmatrix} x_0 + V_{\lambda.1} h_\lambda \\ y_0 + V_{\lambda.2} \varphi_\lambda h_\lambda \\ u_0 + V_{\lambda.3} (\varphi_\lambda h^2_\lambda f + h_\lambda \hat{s}' + \varphi_\lambda h_\lambda \hat{t}') \\ p_0 + V_{\lambda.4} (\varphi_\lambda h_\lambda f + h_\lambda \hat{s}'') \\ q_0 + V_{\lambda.5} (h_\lambda f + \varphi_\lambda h_\lambda \hat{t}'') \end{pmatrix}. \tag{5.3}$$

Let

$$V_{\lambda.j} = v_\lambda \qquad (j = 1, \ldots, 5;\ \lambda = 1, 2, 3), \tag{5.4}$$

and let

$$D_{\lambda 1} = v_\lambda D^*_\lambda \tag{5.5}$$

[2] If necessary the series can be replaced by a truncated series with remainder term. Because of the regularity assumptions on f, enough terms may be kept that the analysis below is not altered.

where

$$D_\lambda^* = \frac{\partial}{\partial\alpha_1} + \varphi_\lambda \frac{\partial}{\partial\alpha_2} + (\varphi_\lambda h_\lambda f + \hat{s}' + \varphi_\lambda \hat{t}') \frac{\partial}{\partial\alpha_3} + \tag{5.6}$$
$$+ (\varphi_\lambda f + \hat{s}'') \frac{\partial}{\partial\alpha_4} + (f + \varphi_\lambda \hat{t}'') \frac{\partial}{\partial\alpha_5}.$$

Then in the Taylor expansion of $f(U_{\lambda 1})$, $h_\lambda D_{\lambda 1}$ plays the part of D in (5.1). Thus

$$f(U_{\lambda 1}) = \sum_{n=0}^{\infty} \frac{h_\lambda^n}{n!} D_{\lambda 1}^n f.$$

From this

$$\boldsymbol{K}_{\lambda 2} = \boldsymbol{H}_\lambda \boldsymbol{\Phi}(\boldsymbol{U}_{\lambda 1}) = \begin{pmatrix} h_\lambda \\ \varphi_\lambda h_\lambda \\ \varphi_\lambda h_\lambda^2 \left[f + h_\lambda D_{\lambda 1} f + \frac{h_\lambda^2}{2} D_{\lambda 1}^2 f + \frac{h_\lambda^3}{3!} D_{\lambda 1}^3 f + \cdots \right] \\ \varphi_\lambda h_\lambda \left[f + h_\lambda D_{\lambda 1} f + \frac{h_\lambda^2}{2} D_{\lambda 1}^2 f + \frac{h_\lambda^3}{3!} D_{\lambda 1}^3 f + \cdots \right] \\ h_\lambda \left[f + h_\lambda D_{\lambda 1} f + \frac{h_\lambda^2}{2} D_{\lambda 1}^2 f + \frac{h_\lambda^3}{3!} D_{\lambda 1}^3 f + \cdots \right] \end{pmatrix}, \tag{5.7}$$

so that

$$\boldsymbol{U}_{\lambda 2} = \boldsymbol{U}_0 + \sum_{i=1}^{2} \boldsymbol{W}_{\lambda i} (\boldsymbol{K}_{\lambda i} + \boldsymbol{B}_\lambda)$$
$$= \begin{pmatrix} x_0 + (W_{\lambda 1.1} + W_{\lambda 2.1}) h_\lambda \\ y_0 + (W_{\lambda 1.2} + W_{\lambda 2.2}) \varphi_\lambda h_\lambda \\ u_0 + W_{\lambda 1.3} K_{\lambda 1.3} + W_{\lambda 2.3} K_{\lambda 2.3} + (W_{\lambda 1.3} + W_{\lambda 2.3}) h_\lambda (\hat{s}' + \varphi_\lambda \hat{t}') \\ p_0 + W_{\lambda 1.4} K_{\lambda 1.4} + W_{\lambda 2.4} K_{\lambda 2.4} + (W_{\lambda 1.4} + W_{\lambda 2.4}) h_\lambda \hat{s}'' \\ q_0 + W_{\lambda 1.5} K_{\lambda 1.5} + W_{\lambda 2.5} K_{\lambda 2.5} + (W_{\lambda 1.5} + W_{\lambda 2.5}) \varphi_\lambda h_\lambda \hat{t}'' \end{pmatrix}. \tag{5.8}$$

Let

$$W_{\lambda 1.j} + W_{\lambda 2.j} = w_\lambda \qquad (j = 1, .., 5;\ \lambda = 1, 2, 3), \tag{5.9}$$

and let

$$D_{\lambda 2} = w_\lambda D_\lambda^*. \tag{5.10}$$

Then, with (5.8) in mind, one can write, using (5.2) and (5.7),

$$w_\lambda h_\lambda \frac{\partial}{\partial\alpha_1} + w_\lambda \varphi_\lambda h_\lambda \frac{\partial}{\partial\alpha_2} + [W_{\lambda 1.3} K_{\lambda 1.3} + W_{\lambda 2.3} K_{\lambda 2.3} + w_\lambda h_\lambda (\hat{s}' + \varphi_\lambda \hat{t}')] \frac{\partial}{\partial\alpha_3} +$$
$$+ [W_{\lambda 1.4} K_{\lambda 1.4} + W_{\lambda 2.4} K_{\lambda 2.4} + w_\lambda h_\lambda \hat{s}''] \frac{\partial}{\partial\alpha_4} +$$
$$+ [W_{\lambda 1.5} K_{\lambda 1.5} + W_{\lambda 2.5} K_{\lambda 2.5} + w_\lambda \varphi_\lambda h_\lambda \hat{t}''] \frac{\partial}{\partial\alpha_5}$$
$$= h_\lambda D_{\lambda 2} + W_{\lambda 2.3} (K_{\lambda 2.3} - \varphi_\lambda h_\lambda^2 f) \frac{\partial}{\partial\alpha_3} +$$
$$+ W_{\lambda 2.4} (K_{\lambda 2.4} - \varphi_\lambda h_\lambda f) \frac{\partial}{\partial\alpha_4} + W_{\lambda 2.5} (K_{\lambda 2.5} - h_\lambda f) \frac{\partial}{\partial\alpha_5}$$
$$= h_\lambda D_{\lambda 2} + W_{\lambda 2.3} \varphi_\lambda h_\lambda^3 \left[D_{\lambda 1} f + \frac{h_\lambda}{2} D_{\lambda 1}^2 f + \frac{h_\lambda^2}{3!} D_{\lambda 1}^3 f + \cdots \right] \frac{\partial}{\partial\alpha_3} +$$
$$+ W_{\lambda 2.4} \varphi_\lambda h_\lambda^2 \left[D_{\lambda 1} f + \frac{h_\lambda}{2} D_{\lambda 1}^2 f + \frac{h_\lambda^2}{3!} D_{\lambda 1}^3 f + \cdots \right] \frac{\partial}{\partial\alpha_4} +$$
$$+ W_{\lambda 2.5} h_\lambda^2 \left[D_{\lambda 1} f + \frac{h_\lambda}{2} D_{\lambda 1}^2 f + \frac{h_\lambda^2}{3!} D_{\lambda 1}^3 + f \cdots \right] \frac{\partial}{\partial\alpha_5}.$$

It is this entire operator which plays the part of D in the general Taylor expansion (5.1) of $f(\boldsymbol{U}_{\lambda 2})$. Thus

$$f(\boldsymbol{U}_{\lambda 2}) = f + h_\lambda D_{\lambda 2} f + \\ + h_\lambda^2 [\tfrac{1}{2} D_{\lambda 2}^2 f + W_{\lambda 2.4}\, \varphi_\lambda D_{\lambda 1} f \cdot f_4 + W_{\lambda 2.5} D_{\lambda 1} f \cdot f_5] + \mathcal{O}(h^3), \tag{5.11}$$

so

$$\boldsymbol{K}_{\lambda 3} = \boldsymbol{H}_\lambda \boldsymbol{\Phi}(\boldsymbol{U}_{\lambda 2}) = \begin{pmatrix} h_\lambda \\ \varphi_\lambda h_\lambda \\ \varphi_\lambda h_\lambda^2 f(\boldsymbol{U}_{\lambda 2}) \\ \varphi_\lambda h_\lambda f(\boldsymbol{U}_{\lambda 2}) \\ h_\lambda f(\boldsymbol{U}_{\lambda 2}) \end{pmatrix}. \tag{5.12}$$

It will be seen in equations (6.1) and (7.1) that the operators $D_{\lambda 1}$ and $D_{\lambda 2}$ are to be compared with derivatives of F. In order to carry out the comparison, let f_i denote $\frac{\partial}{\partial \alpha_i} f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ $(i = 1, \ldots, 5)$. Then

$$F_x = f_1 + f_3 \frac{\partial u}{\partial x} + f_4 \frac{\partial p}{\partial x} + f_5 \frac{\partial q}{\partial x}.$$

The values of interest for the functions in the foregoing computations are those at (x_0, y_0). At the point (x_0, y_0), because of (1.6) and (1.1), one may write

$$F_x = f_1 + s' f_3 + s'' f_4 + f f_5. \tag{5.13}$$

Similarly, at (x_0, y_0),

$$F_y = f_2 + t' f_3 + f f_4 + t'' f_5, \tag{5.14}$$

and

$$F_{xx} = f_{11} + s'^2 f_{33} + s''^2 f_{44} + f^2 f_{55} + \\ + 2(s' f_{13} + s'' f_{14} + f f_{15} + s' s'' f_{34} + s' f f_{35} + s'' f f_{45}) + \\ + s'' f_3 + s''' f_4 + F_x f_5,$$

$$F_{xy} = f_{12} + s' t' f_{33} + s'' f f_{44} + t'' f f_{55} + \\ + t' f_{13} + s' f_{23} + f f_{14} + s'' f_{24} + t'' f_{15} + f f_{25} + \\ + (s' f + t' s'') f_{34} + (s' t'' + t' f) f_{35} + (s'' t'' + f^2) f_{45} + \\ + f f_3 + F_x f_4 + F_y f_5,$$

$$F_{yy} = f_{22} + t'^2 f_{33} + f^2 f_{44} + t''^2 f_{55} + \\ + 2(t' f_{23} + f f_{24} + t'' f_{25} + t' f f_{34} + t' t'' f_{35} + t'' f f_{45}) + \\ + t'' f_3 + F_y f_4 + t''' f_5.$$

(Note particularly the terms in f_3, f_4 and f_5; they are the root cause of the definitions of the $\tilde{s}'$, $\tilde{t}'$, $\tilde{s}''$, $\tilde{t}''$. Their presence here means that, for later purposes, the $\tilde{s}'$, $\tilde{t}'$, $\tilde{s}''$, $\tilde{t}''$ in (5.15) below may not be taken as s', t', s'', t'', respectively; *cf.* equations (6.2), (6.3) and (6.5).)

From the form of (5.13) and (5.14) and the definitions of $D_{\lambda 1}$ and $D_{\lambda 2}$ it is seen that $F_x + \varphi_\lambda F_y$ and $D_\lambda^* f$ should be compared. Indeed

$$F_x + \varphi_\lambda F_y - D_\lambda^* f \\ = [-\varphi_\lambda h_\lambda f - (\hat{s}' - s') - \varphi_\lambda (\hat{t}' - t')] f_3 - (\hat{s}'' - s'') f_4 - (\hat{t}'' - t'') \varphi_\lambda f_5. \tag{5.15}$$

Distinguishing now the meaning of $\sim$ for the cases for u, p and q according to (3.9), one has from (3.11) and (3.12) (still suppressing the superscript) in the case for u:

$$F_x + \varphi_\lambda F_y - D_\lambda^* f = -\left[(\varphi_\lambda h_\lambda f + \tfrac{1}{3} h s'' + \tfrac{1}{3}\varphi_\lambda \varphi h t'') f_3 + \tfrac{1}{3} h s''' f_4 + \tfrac{1}{3}\varphi_\lambda \varphi h t''' f_5\right] + \mathcal{O}(h^2); \tag{5.16}$$

in the case for p:

$$F_x + \varphi_\lambda F_y - D_\lambda^* f = -\left[(\varphi_\lambda h_\lambda f + \tfrac{1}{2} h s'' + \tfrac{1}{3}\varphi_\lambda \varphi h t'') f_3 + \tfrac{1}{2} h s''' f_4 + \tfrac{1}{3}\varphi_\lambda \varphi h t''' f_5\right] + \mathcal{O}(h^2); \tag{5.17}$$

and in the case for q:

$$F_x + \varphi_\lambda F_y - D_\lambda^* f = -\left[(\varphi_\lambda h_\lambda f + \tfrac{1}{3} h s'' + \tfrac{1}{2}\varphi_\lambda \varphi h t'') f_3 + \tfrac{1}{3} h s''' f_4 + \tfrac{1}{2}\varphi_\lambda \varphi h t''' f_5\right] + \mathcal{O}(h^2). \tag{5.18}$$

To have the coefficients $\frac{1}{2}$ and $\frac{1}{3}$ appear as shown was the object of introducing the quantities $\tilde{s}'$, $\tilde{s}''$, $\tilde{t}'$ and $\tilde{t}''$ into the $\boldsymbol{B}_\lambda^\omega$. The reason for having these particular coefficients is noted at the end of § 6.

The above equations (5.16), (5.17) and (5.18) each represent three equations, one for each of $\lambda = 1, 2, 3$. Later, we shall wish to multiply the equations (5.16) each by a certain α_λ and add, and to do the same for (5.17) and (5.18).

Note further that

$$\frac{1}{v_\lambda^2} D_{\lambda 1}^2 = \frac{1}{w_\lambda^2} D_{\lambda 2}^2 = D_\lambda^{*2} \tag{5.19}$$

because the coefficients of the $\partial/\partial \alpha_i$ in D_λ^* are *constants*. Thus, for the second order operators, the comparison is between D_λ^{*2} and $F_{xx} + 2\varphi_\lambda F_{xy} + \varphi_\lambda^2 F_{yy}$ (*cf.* equation (6.1)). One finds that

$$\begin{aligned} D_\lambda^{*2} f = f_{11} &+ 2\varphi_\lambda f_{12} + \varphi_\lambda^2 f_{22} + \\ &+ (\varphi_\lambda h_\lambda f + \hat{s}' + \varphi_\lambda \hat{t}')^2 f_{33} + (\varphi_\lambda f + \hat{s}'')^2 f_{44} + (f + \varphi_\lambda \hat{t}'')^2 f_{55} + \\ &+ 2\left[(\varphi_\lambda h_\lambda f + \hat{s}' + \varphi_\lambda \hat{t}') f_{13} + (\varphi_\lambda f + \hat{s}'') f_{14} + (f + \varphi_\lambda \hat{t}'') f_{15} + \right. \\ &+ \varphi_\lambda(\varphi_\lambda h_\lambda f + \hat{s}' + \varphi_\lambda \hat{t}') f_{23} + \varphi_\lambda(\varphi_\lambda f + \hat{s}'') f_{24} + \varphi_\lambda(f + \varphi_\lambda \hat{t}'') f_{25} + \\ &+ (\varphi_\lambda h_\lambda f + \hat{s}' + \varphi_\lambda \hat{t}')(\varphi_\lambda f + \hat{s}'') f_{34} + (\varphi_\lambda h_\lambda f + \hat{s}' + \varphi_\lambda \hat{t}')(f + \varphi_\lambda \hat{t}'') f_{35} + \\ &\left. + (\varphi_\lambda f + \hat{s}'')(f + \varphi_\lambda \hat{t}'') f_{45}\right]. \end{aligned}$$

For the comparison just mentioned it will be sufficient in the sequel for terms merely to agree within order h. Thus, recalling that $\hat{s}' = s' + \mathcal{O}(h)$, *etc.*, one has, for example, for the coefficients of f_{33}, that

$$(s' + \varphi_\lambda t')^2 = (\varphi_\lambda h_\lambda f + \hat{s}' + \varphi_\lambda \hat{t}')^2 + \mathcal{O}(h),$$

and hence

$$\begin{aligned} F_{xx} &+ 2\varphi_\lambda F_{xy} + \varphi_\lambda^2 F_{yy} - D_\lambda^{*2} f \\ &= (s'' + 2\varphi_\lambda f + \varphi_\lambda^2 t'') f_3 + (s''' + 2\varphi_\lambda F_x + \varphi_\lambda^2 F_y) f_4 + \\ &\quad + (F_x + 2\varphi_\lambda F_y + \varphi_\lambda^2 t''') f_5 + \mathcal{O}(h). \end{aligned} \tag{5.20}$$

As with (5.16), this equation stands for three equations: $\lambda=1, 2, 3$. Of the coefficients of f_3, f_4 and f_5 here, those parts involving s'', t'', s''', t''' are similar to those obtained in (5.16—18). It is this fact which, through the definitions of the $\boldsymbol{B}_\lambda^\omega$, allows the complete matching of terms in the next sections, as in the computation of (6.5).

§ 6. Requirements on the Parameters in the Case for u

In this and the succeeding two sections, requirements on the parameters analogous to those in R-K 1 will be derived. Since in this section only one case (*viz* u) is considered, the identifying superscript will usually be suppressed.

From the equations (4.1), (4.2) and (4.3) defining the computational procedure, one has

$$\Delta_{.3}=\Bigg(\sum_{\substack{\lambda=1,2,3\\ i=1,2,3}} A_{\lambda i.3}\, K^u_{\lambda i.3}\Bigg)+B_{.3}.$$

In $\Delta_{.3}$ the quantities $K^u_{\lambda i.3}$ and $B_{.3}$ are to be replaced by their equivalents as given by (5.2), (5.7) and (5.12) (and (5.11)) and the definition (3.4). Here, the quantities Δs and Δt involved in $B_{.3}$ should be replaced by their Taylor expansions. The terms of $\Delta_{.3}$ are to be grouped according to powers of h and compared (by subtraction) with the corresponding terms in the Taylor series expansion (2.1) of Δu. The terms of (2.1) involving s and t are exactly those terms in the Taylor expansions of Δs and Δt which appear in $\Delta_{.3}$ because of $B_{.3}$. Hence, all the terms in s and t cancel when $\Delta_{.3}$ is subtracted from Δu. The remaining terms of $\Delta_{.3}$ involve the formulas for the $K_{\lambda i.3}$. In these it is to be recalled that

$$\varphi_\lambda=e_\lambda\,\varphi, \qquad h_\lambda=g_\lambda\,h,$$

so that φ and h can often be factored out of a given term. Thus, letting $\varepsilon_{.3}$ denote the absolute difference between Δu and $\Delta_{.3}$, one finds

$$\begin{aligned}
\varepsilon_{.3} &= |\Delta u-\Delta_{.3}|\\
&=\Bigg|\varphi h^2 f\Big[1-\sum_{\lambda,i} A_{\lambda i.3}\, e_\lambda g_\lambda^2\Big]+\\
&\quad+\varphi h^3\Big[\tfrac12 F_x+\tfrac12\varphi F_y-\sum_{\lambda=1}^{3}(A_{\lambda 2.3}\, e_\lambda g_\lambda^3 D_{\lambda 1} f+A_{\lambda 3.3}\, e_\lambda g_\lambda^3 D_{\lambda 2} f)\Big]+\\
&\quad+\varphi h^4\Big[\tfrac16 F_{xx}+\tfrac14\varphi F_{xy}+\tfrac16\varphi^2 F_{yy}-\sum_{\lambda=1}^{3}(A_{\lambda 2.3}\, e_\lambda g_\lambda^4\cdot\tfrac12 D_{\lambda 1}^2 f+A_{\lambda 3.3}\, e_\lambda g_\lambda^4\cdot\tfrac12 D_{\lambda 2}^2 f)-\\
&\quad-\sum_{\lambda=1}^{3} A_{\lambda 3.3}\, e_\lambda g_\lambda^4 (W_{\lambda 2.4}\,\varphi_\lambda D_{\lambda 1} f\cdot f_4+W_{\lambda 2.5} D_{\lambda 1} f\cdot f_5)\Big]\Bigg|+\mathcal{O}(h^5).
\end{aligned} \tag{6.1}$$

The object is now to find requirements on the parameters such that, if the requirements are satisfied, then $\varepsilon_{.3}$ is of order h^5. This will be done by demanding that the coefficients of the lower powers of h in (6.1) be zero. The coefficients will be treated in turn. First, to eliminate the h^2-term requires

$$\sum_{\substack{\lambda=1,2,3\\ i=1,2,3}} A_{\lambda i.3}\, e_\lambda g_\lambda^2=1. \tag{R 1 u}$$

For the coefficient of h^3, recall that $\frac{1}{v_\lambda} D_{\lambda 1} = \frac{1}{w_\lambda} D_{\lambda 2} = D^*_\lambda$. Thus D^*_λ is a factor of each term of the summation in the coefficient of h^3. The coefficient of D^*_λ is taken as the definition of an "intermediate" parameter $\alpha_{\lambda.3}$:

$$(\mathrm{A}_\lambda) \qquad \alpha_{\lambda.3} = e_\lambda g_\lambda^3 (A_{\lambda 2.3} v_\lambda + A_{\lambda 3.3} w_\lambda).$$

Now require

$$(\mathrm{R}\,2\,\mathrm{u}) \qquad \sum_{\lambda=1}^{3} \alpha_{\lambda.3} = \tfrac{1}{2},$$

$$(\mathrm{R}\,3\,\mathrm{u}) \qquad \sum_{\lambda=1}^{3} \alpha_{\lambda.3} e_\lambda = \tfrac{1}{2}.$$

Note that the coefficient of F_y involves φ in the h^3-term of (6.1), while in (5.16) it involves φ_λ. This is the reason for the e_λ which appear in (R 3 u). Multiplying the λ^{th} equation (5.16) by $\alpha_{\lambda.3}$, adding the resulting three equations for $\lambda = 1, 2, 3$, and making use of (R 2u) and (R 3u), one thus sees that the coefficient of φh^3 in (6.1) is given by

$$\begin{aligned}
&\tfrac{1}{2} F_x + \tfrac{1}{2} \varphi F_y - \sum_{\lambda=1}^{3} \alpha_{\lambda.3} D^*_\lambda f \\
(6.2) \qquad &= -\Big[\Big(\sum_\lambda \alpha_{\lambda.3} \varphi_\lambda h_\lambda f + \tfrac{1}{2}\cdot\tfrac{1}{3} h s'' + \tfrac{1}{2}\cdot\tfrac{1}{3}\varphi^2 h t''\Big) f_3 + \\
&\quad + \tfrac{1}{2}\cdot\tfrac{1}{3} h s''' f_4 + \tfrac{1}{2}\cdot\tfrac{1}{3}\varphi^2 h t''' f_5\Big] + \mathcal{O}(h^2).
\end{aligned}$$

The right member of (6.2) has h as a factor; since the member is already part of the h^3-term, the coefficient of h thus found here can be moved into the coefficient of h^4 in (6.1). The $\mathcal{O}(h^2)$ term in (6.2), when multiplied by h^3, can be incorporated into the fifth-order term of $\varepsilon_{.3}$.

Hence, the coefficient of φh^4 in (6.1), including the part from the h^3-term, is

$$\begin{aligned}
&\Big\{\tfrac{1}{6} F_{xx} + \tfrac{1}{4}\varphi F_{xy} + \tfrac{1}{6}\varphi^2 F_{yy} - \sum_\lambda \tfrac{1}{2} e_\lambda g_\lambda^4 (A_{\lambda 2.3} v_\lambda^2 + A_{\lambda 3.3} w_\lambda^2) D^{*2}_\lambda f\Big\} - \\
(6.3) \qquad &- \sum_\lambda A_{\lambda 3 3} e_\lambda g_\lambda^4 (W_{\lambda 2.4} \varphi_\lambda D_{\lambda 1} f \cdot f_4 + W_{\lambda 2.5} D_{\lambda 1} f \cdot f_5) - \\
&- \Big[\Big(\sum_\lambda \alpha_{\lambda.3} \varphi_\lambda g_\lambda f + \tfrac{1}{6} s'' + \tfrac{1}{6}\varphi^2 t''\Big) f_3 + \tfrac{1}{6} s''' f_4 + \tfrac{1}{6}\varphi^2 t''' f_5\Big].
\end{aligned}$$

Let

$$(\mathrm{B}_\lambda) \qquad \beta_{\lambda.3} = \tfrac{1}{2} e_\lambda g_\lambda^4 (A_{\lambda 2.3} v_\lambda^2 + A_{\lambda 3.3} w_\lambda^2),$$

and require

$$(\mathrm{R}\,4\,\mathrm{u}) \qquad \sum_\lambda \beta_{\lambda 3} = \tfrac{1}{6},$$

$$(\mathrm{R}\,5\,\mathrm{u}) \qquad \sum_\lambda \beta_{\lambda.3} e_\lambda = \tfrac{1}{8},$$

$$(\mathrm{R}\,6\,\mathrm{u}) \qquad \sum_\lambda \beta_{\lambda.3} e_\lambda^2 = \tfrac{1}{6}.$$

Note that the coefficients of F_{xy} and F_{yy} in (6.3) involve φ, while in (5.20) they involve φ_λ. This is the reason for the e_λ and e_λ^2 which appear in (R 5u) and (R 6u). Multiplying the λ^{th} equation (5.20) by $\beta_{\lambda.3}$, adding the three equations

for $\lambda = 1, 2, 3$ and making use of (R 4u), (R 5u) and (R 6u), one has for the part of (6.3) in curly braces

$$\begin{aligned}(6.4)\qquad & \tfrac{1}{6} F_{xx} + \tfrac{2}{8} \varphi F_{\lambda y} + \tfrac{1}{6} \varphi^2 F_{yy} - \sum_\lambda \beta_{\lambda.3} D_\lambda^{*2} f \\ & = \left(\tfrac{1}{6} s'' + \tfrac{1}{4} \varphi f + \tfrac{1}{6} \varphi^2 t''\right) f_3 + \left(\tfrac{1}{6} s''' + \tfrac{1}{4} \varphi F_x + \tfrac{1}{6} \varphi^2 F_y\right) f_4 + \\ & + \left(\tfrac{1}{6} F_x + \tfrac{1}{4} \varphi F_y + \tfrac{1}{6} \varphi^2 t'''\right) f_5 + \mathcal{O}(h).\end{aligned}$$

Since the $\mathcal{O}(h)$ term in (6.4) can be incorporated into the fifth-order term of $\varepsilon_{.3}$, (6.3) becomes

$$\begin{aligned}(6.5)\qquad & \left(\tfrac{1}{4} \varphi f - \sum_\lambda \alpha_{\lambda.3} \varphi_\lambda g_\lambda f\right) f_3 + \left(\tfrac{1}{4} \varphi F_x + \tfrac{1}{6} \varphi^2 F_y\right) f_4 + \left(\tfrac{1}{6} F_x + \tfrac{1}{4} \varphi F_y\right) f_5 - \\ & - \sum_\lambda A_{\lambda 3.3} e_\lambda g_\lambda^4 \left(W_{\lambda 2.4} \varphi_\lambda D_{\lambda 1} f \cdot f_4 + W_{\lambda 2.5} D_{\lambda 1} f \cdot f_5\right).\end{aligned}$$

(Note the elimination of the terms in s'', t'', s''' and t'''.)

Consider the coefficients of f_4 in (6.5). Let

$$(\mathrm{C}_\lambda)\qquad \gamma_{\lambda.3} = e_\lambda^2 g_\lambda^4 A_{\lambda 3.3} W_{\lambda 2.4} v_\lambda$$

and require

$$(\mathrm{R\ 7\ u})\qquad \sum_\lambda \gamma_{\lambda.3} = \tfrac{1}{4},$$

$$(\mathrm{R\ 8\ u})\qquad \sum_\lambda \gamma_{\lambda.3} e_\lambda = \tfrac{1}{6}.$$

Then from these requirements and from equations (5.16) it follows (in the same manner as for equation (6.2)) that

$$\varphi\left(\tfrac{1}{4} F_x + \tfrac{1}{6} \varphi F_y - \sum_\lambda \gamma_{\lambda.3} D_\lambda^* f\right) f_4 = \mathcal{O}(h).$$

Thus, this difference can be incorporated into the fifth-order term of $\varepsilon_{.3}$.

Consider the coefficients of f_5 in (6.5). Let

$$(\mathrm{D}_\lambda)\qquad \delta_{\lambda.3} = e_\lambda g_\lambda^4 A_{\lambda 3.3} W_{\lambda 2.5} v_\lambda$$

and require

$$(\mathrm{R\ 9\ u})\qquad \sum_\lambda \delta_{\lambda.3} = \tfrac{1}{6},$$

$$(\mathrm{R\ 10\ u})\qquad \sum_\lambda \delta_{\lambda.3} e_\lambda = \tfrac{1}{4}.$$

Then, again, by adding equations (5.16) each multiplied by the appropriate $\delta_{\lambda.3}$, one finds

$$\left(\tfrac{1}{6} F_x + \tfrac{1}{4} \varphi F_y - \sum_\lambda \delta_{\lambda.3} D_\lambda^* f\right) f_5 = \mathcal{O}(h)$$

which can be incorporated into the fifth-order term of $\varepsilon_{.3}$.

Finally, so that the coefficient of f_3 in (6.5) becomes zero, require

$$(\mathrm{R\ 11\ u})\qquad \sum_\lambda \alpha_{\lambda.3} e_\lambda g_\lambda = \tfrac{1}{4}.$$

If all the above requirements (R 1u)—(R 11u) are satisfied, then it is seen that $\varepsilon_{.3} = \mathcal{O}(h^5)$, as desired.

It is worth noting that the particular coefficients $\frac{1}{6}$ appearing in the right member of (6.2) were originally set up by the use of $\tilde{s}'$, $\tilde{s}''$, $\tilde{t}'$ and $\tilde{t}''$ in the definition of $\boldsymbol{B}^u_\lambda$. This was, of course, the reason for so defining $\boldsymbol{B}^u_\lambda$. In the cases for p and q still to be considered, the same reason will appear for the definitions of $\boldsymbol{B}^p_\lambda$ and $\boldsymbol{B}^q_\lambda$.

§ 7. Requirements on the Parameters in the Case for p

The derivation of the requirements in the case for p is, with two exceptions, exactly the same as in the case for u. The first exception is that the power of h is lower by one in the various terms to be matched in $\Delta_{.4}$ and the Taylor expansion of Δp. The second exception is that equations (5.17) are used instead of (5.16). Because the cases are so much alike, most of the explanatory comments will be omitted here.

From (4.1), (4.2) and (4.3),

$$\Delta_{.4} = \Bigl(\sum_{\substack{\lambda=1,2,3\\ i\ 1,2,3}} A_{\lambda i.4} K^p_{\lambda i.4}\Bigr) + B_{.4}.$$

Substituting into this the formulas from § 5 for $K_{\lambda i.4}$ and the definition of $B_{.4}$, one finds

$$\begin{aligned}
\varepsilon_{.4} &= |\Delta p - \Delta_{.4}| \\
&= \Bigl|\varphi h f \Bigl[1 - \sum_{\lambda, i} A_{\lambda i.4} e_\lambda g_\lambda\Bigr] + \\
&\quad + \varphi h^2 \Bigl[F_x + \tfrac{1}{2}\varphi F_y - \sum_\lambda (A_{\lambda 2.4} e_\lambda g_\lambda^2 D_{\lambda 1} f + A_{\lambda 3.4} e_\lambda g_\lambda^2 D_{\lambda 2} f)\Bigr] + \qquad (7.1)\\
&\quad + \varphi h^3 \Bigl[\tfrac{1}{2} F_{xx} + \tfrac{1}{2}\varphi F_{xy} + \tfrac{1}{6}\varphi^2 F_{yy} - \sum_\lambda (A_{\lambda 2.4} e_\lambda g_\lambda^3 \cdot \tfrac{1}{2} D_{\lambda 1}^2 f + A_{\lambda 3.4} e_\lambda g_\lambda^3 \cdot \tfrac{1}{2} D_{\lambda 2}^2 f) - \\
&\quad - \sum_\lambda A_{\lambda 3.4} e_\lambda g_\lambda^3 (W_{\lambda 2.4}\varphi_\lambda D_{\lambda 1} f \cdot f_4 + W_{\lambda 2.5} D_{\lambda 1} f \cdot f_5)\Bigr]\Bigr| + \mathcal{O}(h^4).
\end{aligned}$$

To make the coefficient of $\varphi h f$ vanish, require

(R 1 p) $$\sum_{\substack{\lambda=1,2,3\\ i=1,2,3}} A_{\lambda i.4} e_\lambda g_\lambda = 1.$$

Next, set

(A_λ) $$\alpha_{\lambda.4} = e_\lambda g_\lambda^2 (A_{\lambda 2.4} v_\lambda + A_{\lambda 3.4} w_\lambda),$$

and require

(R 2 p) $$\sum_\lambda \alpha_{\lambda.4} = 1,$$

(R 3 p) $$\sum_\lambda \alpha_{\lambda.4} e_\lambda = \tfrac{1}{2}.$$

From equations (5.17), the coefficient of φh^2 in (7.1) becomes then

$$\begin{aligned}
&F_x + \tfrac{1}{2}\varphi F_y - \sum_\lambda \alpha_{\lambda.4} D^*_\lambda f \\
(7.2)\qquad &= -\Bigl[\Bigl(\sum_\lambda \alpha_{\lambda.4}\varphi_\lambda h_\lambda f + \tfrac{1}{2} h s'' + \tfrac{1}{2}\cdot\tfrac{1}{3}\varphi^2 h t''\Bigr) f_3 + \\
&\quad + \tfrac{1}{2} h s''' f_4 + \tfrac{1}{2}\cdot\tfrac{1}{3}\varphi^2 h t''' f_5\Bigr] + \mathcal{O}(h^2).
\end{aligned}$$

Note that h is a factor of the right-hand side. Thus the coefficient of φh^3 in (7.1), with the additional terms from (7.2) included, is now

$$\begin{aligned}(7.3)\qquad &\left\{\tfrac{1}{2}F_{xx}+\tfrac{1}{2}\varphi F_{xy}+\tfrac{1}{6}\varphi^2 F_{yy}-\sum_\lambda \tfrac{1}{2}e_\lambda g_\lambda^3(A_{\lambda 2.4}v_\lambda^2+A_{\lambda 3.4}w_\lambda^2)D_\lambda^{*2}f\right\}-\\ &-\sum_\lambda A_{\lambda 3.4}e_\lambda g_\lambda^3(W_{\lambda 2.4}\varphi_\lambda D_{\lambda 1}f\cdot f_4+W_{\lambda 2.5}D_{\lambda 1}f\cdot f_5)-\\ &-\left[\left(\sum_\lambda \alpha_{\lambda .4}\varphi_\lambda g_\lambda f+\tfrac{1}{2}s''+\tfrac{1}{6}\varphi^2 t''\right)f_3+\tfrac{1}{2}s'''f_4+\tfrac{1}{6}\varphi^2 t'''f_5\right].\end{aligned}$$

Let

$$(\mathrm{B}_\lambda)\qquad \beta_{\lambda .4}=\tfrac{1}{2}e_\lambda g_\lambda^3(A_{\lambda 2 4}v_\lambda^2+A_{\lambda 3.4}w_\lambda^2),$$

and set

$$(\mathrm{R\,4\,p})\qquad \sum_\lambda \beta_{\lambda .4}=\tfrac{1}{2},$$

$$(\mathrm{R\,5\,p})\qquad \sum_\lambda \beta_{\lambda .4}e_\lambda=\tfrac{1}{4},$$

$$(\mathrm{R\,6\,p})\qquad \sum_\lambda \beta_{\lambda .4}e_\lambda^2=\tfrac{1}{6}.$$

From (5.20), the part of (7.3) in curly braces becomes

$$\begin{aligned}(7.4)\qquad &\tfrac{1}{2}F_{xx}+\tfrac{2}{4}\varphi F_{xy}+\tfrac{1}{6}\varphi^2F_{yy}-\sum_\lambda \beta_{\lambda .4}D_\lambda^{*2}f\\ &=(\tfrac{1}{2}s''+\tfrac{1}{2}\varphi f+\tfrac{1}{6}\varphi^2 t'')f_3+(\tfrac{1}{2}s'''+\tfrac{1}{2}\varphi F_x+\tfrac{1}{6}\varphi^2F_y)f_4+\\ &+(\tfrac{1}{2}F_x+\tfrac{1}{2}\varphi F_y+\tfrac{1}{6}\varphi^2 t''')f_5+\mathcal{O}(h).\end{aligned}$$

Then, from this, (7.3) becomes (since the $\mathcal{O}(h)$ term can be incorporated into the fourth-order term of $\varepsilon_{.4}$)

$$\begin{aligned}(7.5)\qquad &\left(\tfrac{1}{2}\varphi f-\sum_\lambda \alpha_{\lambda .4}\varphi_\lambda g_\lambda f\right)f_3+(\tfrac{1}{2}\varphi F_x+\tfrac{1}{6}\varphi^2F_y)f_4+(\tfrac{1}{2}F_x+\tfrac{1}{2}\varphi F_y)f_5-\\ &-\sum_\lambda A_{\lambda 3.4}e_\lambda g_\lambda^3(W_{\lambda 2.4}\varphi_\lambda D_{\lambda 1}f\cdot f_4+W_{\lambda 2.5}D_{\lambda 1}f\cdot f_5).\end{aligned}$$

Consider the coefficients of f_4. Let

$$(\mathrm{C}_\lambda)\qquad \gamma_{\lambda .4}=e_\lambda^2 g_\lambda^3 A_{\lambda 3.4}W_{\lambda 2.4}v_\lambda$$

and require

$$(\mathrm{R\,7\,p})\qquad \sum_\lambda \gamma_{\lambda .4}=\tfrac{1}{2},$$

$$(\mathrm{R\,8\,p})\qquad \sum_\lambda \gamma_{\lambda .4}e_\lambda=\tfrac{1}{6}.$$

From these conditions and from equations (5.17) it follows that

$$\varphi\left(\tfrac{1}{2}F_x+\tfrac{1}{6}\varphi F_y-\sum_\lambda \gamma_{\lambda .4}D_\lambda^* f\right)f_4=\mathcal{O}(h).$$

Thus, this difference can be incorporated into the fourth-order term of $\varepsilon_{.4}$.

Consider the coefficients of f_5 in (7.5). Let

$$(\mathrm{D}_\lambda)\qquad \delta_{\lambda .4}=e_\lambda g_\lambda^3 A_{\lambda 3 4}W_{\lambda 2.5}v_\lambda$$

and require

(R 9 p) $$\sum_\lambda \delta_{\lambda.4} = \tfrac{1}{2},$$

(R 10 p) $$\sum_\lambda \delta_{\lambda.4} e_\lambda = \tfrac{1}{2}.$$

From these conditions and from equations (5.17) it follows that

$$\left(\tfrac{1}{2} F_x + \tfrac{1}{2}\varphi F_y - \sum_\lambda \delta_{\lambda.4} D_\lambda^* f\right) f_5 = \mathcal{O}(h).$$

This difference can be incorporated into the fourth-order term of $\varepsilon_{.4}$.

Finally, so that the coefficient of f_3 in (7.5) becomes zero, require

(R 11 p) $$\sum_\lambda \alpha_{\lambda.4} e_\lambda g_\lambda = \tfrac{1}{2}.$$

If the parameters satisfy the above requirements, then $\varepsilon_{.4} = \mathcal{O}(h^4)$ as desired.

§ 8. Requirements on the Parameters in the Case for *q*

The procedure for this case is the same as before. For this reason only the resulting requirements are given here:

$$\sum_{\substack{\lambda=1,2,3\\ i=1,2,3}} A_{\lambda i.5}\, g_\lambda = 1;$$

$$\alpha_{\lambda.5} = g_\lambda^2 (A_{\lambda 2.5} v_\lambda + A_{\lambda 3.5} w_\lambda),$$

$$\sum_\lambda \alpha_{\lambda.5} = \tfrac{1}{2},$$

$$\sum_\lambda \alpha_{\lambda.5} e_\lambda = 1;$$

$$\beta_{\lambda.5} = \tfrac{1}{2} g_\lambda^3 (A_{\lambda 2.5} v_\lambda^2 + A_{\lambda 3.5} w_\lambda^2),$$

$$\sum_\lambda \beta_{\lambda.5} = \tfrac{1}{6},$$

$$\sum_\lambda \beta_{\lambda.5} e_\lambda = \tfrac{1}{4},$$

$$\sum_\lambda \beta_{\lambda.5} e_\lambda^2 = \tfrac{1}{2};$$

$$\gamma_{\lambda.5} = e_\lambda g_\lambda^3 A_{\lambda 3.5} W_{\lambda 2.4} v_\lambda,$$

$$\sum_\lambda \gamma_{\lambda.5} = \tfrac{1}{2},$$

$$\sum_\lambda \gamma_{\lambda.5} e_\lambda = \tfrac{1}{2};$$

$$\delta_{\lambda.5} = g_\lambda^3 A_{\lambda 3.5} W_{\lambda 2.5} v_\lambda,$$

$$\sum_\lambda \delta_{\lambda.5} = \tfrac{1}{6},$$

$$\sum_\lambda \delta_{\lambda.5} e_\lambda = \tfrac{1}{2};$$

$$\sum_\lambda \alpha_{\lambda.5} e_\lambda g_\lambda = \tfrac{1}{2}.$$

§ 9. Further Requirements

Besides the conditions on the parameters imposed in the preceding sections there are also the natural requirements (see (3.7))

$$0 < g_\lambda^\omega \leqq 1, \quad 0 < e_\lambda^\omega g_\lambda^\omega \leqq 1.$$

It is also reasonable to impose simple requirements on the e_λ and g_λ which yield some kind of symmetry among the subrectangles with dimensions h_λ by $\varphi_\lambda h_\lambda$. Each of these subrectangles has one corner at (x_0, y_0) and the diagonally opposite corner at $(x_0 + h_\lambda, y_0 + \varphi_\lambda h_\lambda)$. The symmetry requirements which seem most natural are that for each case, u, p, and q, (1) the first rectangle ($\lambda = 1$) have the corner $(x_0 + h_1, y_0 + \varphi_1 h_1)$ on the diagonal of the basic subrectangle (see Figure 1), and, (2) the corners $(x_0 + h_2, y_0 + \varphi_2 h_2)$ and $(x_0 + h_3, y_0 + \varphi_3 h_3)$ of the second and third subrectangles be symmetrically placed (as in the Figure) with respect to this diagonal. This symmetry is expressed by the proportions

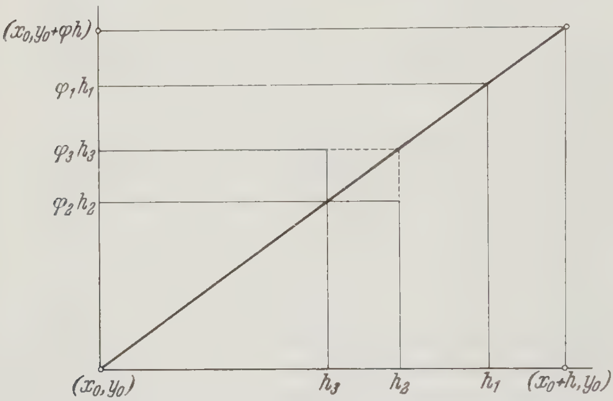

Fig. 1

$$\frac{\varphi h}{h} = \frac{\varphi_1 h_1}{h_1} = \frac{\varphi_2 h_2}{h_3} = \frac{\varphi_3 h_3}{h_2}.$$

From these it follows that $e_1 = 1$, $e_3 = 1/e_2$ and $g_3 = e_2 g_2$, with e_2, g_1 and g_2 still free.

Further, in consideration of the manner in which $\boldsymbol{V}_\lambda$ and $\boldsymbol{W}_{\lambda i}$ enter (*cf.* (4.1)), it is only natural to require

$$v_\lambda \geqq 0 \quad \text{and} \quad w_\lambda = W_{\lambda 1.j} + W_{\lambda 2.j} \geqq 0 \qquad (\lambda = 1, 2, 3;\ j = 1, \ldots, 5).$$

This simply says that when an increment $\boldsymbol{K}_{\lambda i}$ for $\boldsymbol{U}$ is found, $\boldsymbol{U}_0$ is altered "in the direction of" $\boldsymbol{K}_{\lambda i}$. Note that $w_\lambda \geqq 0$ does not imply that both $W_{\lambda 1.j}$ and $W_{\lambda 2.j}$ are non-negative. Indeed, in the system of parameter values given in the next section, $W_{21.4}^u = -1$ and $W_{22.4}^u = 2$ so that $w_2^u = 1$. This is reasonable to permit, for, in (4.1), this means that $U_{22.4}^u$ is given by

$$U_{22.4}^u = p_0 + K_{22.4}^u + (K_{22.4}^u - K_{21.4}^u) + B_{.4}.$$

Thus p_0 is altered principally by the amount $K_{22.4}^u$, with an additional alteration equal to the amount the second increment $K_{22.4}^u$ exceeds the first increment $K_{21.4}^u$. (This is reminiscent of the over-relaxation technique common in numerical work.)

The determination of the parameter values of § 10 was carried out under the further requirement that all the $A_{\lambda i.j}$ be non-negative. An argument such as that just given would imply that this requirement is perhaps too stringent.

From the preceding extra requirements that v_λ, w_λ and $A_{\lambda i.j}$ be non-negative it follows that the intermediate parameters $\alpha_{\lambda.j}$ and $\beta_{\lambda.j}$ are non-negative. In

carrying out the actual determination of a system of parameters, it was also assumed that $\gamma_{\lambda.j}$ and $\delta_{\lambda.j}$ were to be non-negative.

It is seen from the preceding sections that for each case, u, p and q, there are 27 unknown parameters and 11 requirements. The additional requirements just discussed reduce somewhat the apparent underdeterminateness of the system. However, the non-linearity of the equations still makes study of the system difficult.

An important question for the future improvement of the procedure is pertinent here. With the parameter values given in §10, f is evaluated fifteen times for each subrectangle. Could values for e_λ, g_λ, v_λ, w_λ, $W_{\lambda 2.4}$ and $W_{\lambda 2.5}$ be found which reduce the number of points of evaluation of f? With regard to the possible relaxation of some of the extra requirements above, the manner in which the parameters enter in the convergence proof in §12 should be considered.

§10. Values for the Parameters

To obtain the following system of parameter values, all of the requirements of the preceding sections were imposed. It was found that, in order to have the same values of e_λ used in all three cases, u, p and q, and still yield non-negative solutions of (R 4)—(R 10), it was necessary that $e_2=\frac{1}{3}$ and $e_3=3$ (or *vice versa*). This determined separate values $\beta^{\omega}_{\lambda.j}$ for each of the three cases. The definitions of $\alpha_{\lambda.j}$ and $\beta_{\lambda.j}$ (as well as of $D_{\lambda 1}$ and $D_{\lambda 2}$) suggested choosing $v_\lambda=w_\lambda=1$ for all three cases. These definitions, with the known values of $\beta^{\omega}_{\lambda.j}$, and equations (R 2), (R 3) led to fixing the values of g^{ω}_λ. From equations (R 7) to (R 10), $\gamma^{\omega}_{\lambda.j}$ and $\delta^{\omega}_{\lambda.j}$ could then be determined. The preceding choices and the interrelation between the $\alpha_{\lambda.j}$ and $\beta_{\lambda.j}$ already insured the satisfaction of (R 11). In (R 1), values $A_{\lambda 1.j}$ were still free to be chosen to satisfy this requirement.

Throughout the above analysis, a preliminary study of the system

$$a+b+c=d_1,$$

$$a+b\,e+\frac{c}{e}=d_2,$$

$$a+b\,e^2+\frac{c}{e^2}=d_3$$

was of considerable aid. The solutions a, b, c of this system can be obtained explicitly in terms of e, d_1, d_2 and d_3. Equations (R 4), (R 5) and (R 6) form a system of this type, while (R 7), (R 8) and (R 9), (R 10) form related systems

$$b+c=d_1-a,$$

$$b\,e+\frac{c}{e}=d_2-a.$$

Here, the first unknown, a, is suggested as a parameter in this system.

Let $\boldsymbol{O}$ and $\boldsymbol{I}$ denote the zero and unit matrices, respectively. The system of values obtained using the analysis outlined above is the following:

In all three cases, $e_1=1$, $e_2=\frac{1}{3}$, $e_3=3$ and $\boldsymbol{V}_1=\boldsymbol{V}_2=\boldsymbol{V}_3=\boldsymbol{I}$.

For the computations for u:

$$
\begin{array}{lll}
g_1^u = 1 & g_2^u = \dfrac{2}{3} & g_3^u = \dfrac{2}{9} \\
\boldsymbol{W}_{11}^u = \boldsymbol{O} & \boldsymbol{W}_{21}^u = \operatorname{diag}(1, 1, 1, -1, 1) & \boldsymbol{W}_{31}^u = \operatorname{diag}(1, 1, 1, 1, -1) \\
\boldsymbol{W}_{12}^u = \boldsymbol{I} & \boldsymbol{W}_{22}^u = \operatorname{diag}(0, 0, 0, 2, 0) & \boldsymbol{W}_{32}^u = \operatorname{diag}(0, 0, 0, 0, 2) \\
A_{11.3} = \dfrac{1}{32} & A_{21.3} = 0 & A_{31.3} = 0 \\
A_{12.3} = 0 & A_{22.3} = 0 & A_{32.3} = 0 \\
A_{13.3} = \dfrac{1}{8} & A_{23.3} = \dfrac{3^6}{2^8} & A_{33.3} = \dfrac{3^6}{2^8}.
\end{array}
$$

For the computations for p:

$$
\begin{array}{lll}
g_1^p = 1 & g_2^p = 1 & g_3^p = \frac{1}{3} \\
\boldsymbol{W}_{11}^p = \boldsymbol{W}_{31}^u & \boldsymbol{W}_{21}^p = \boldsymbol{W}_{21}^u & \\
\boldsymbol{W}_{12}^p = \boldsymbol{W}_{32}^u & \boldsymbol{W}_{22}^p = \boldsymbol{W}_{22}^u & \\
A_{13.4} = \frac{1}{4} & A_{23.4} = \frac{9}{4}. &
\end{array}
$$

All other $A_{\lambda i.4}$ are zero.

Note that no computations with $\lambda = 3$ are needed in the case for p because $A_{3i.4} = 0$ $(i = 1, 2, 3)$. Hence, $\boldsymbol{W}_{31}^p$ and $\boldsymbol{W}_{32}^p$ are not given above.

For the computations for q:

$$
\begin{array}{lll}
g_1^q = 1 & g_2^q = 1 & g_3^q = \frac{1}{3} \\
\boldsymbol{W}_{11}^q = \boldsymbol{W}_{21}^u & & \boldsymbol{W}_{31}^q = \boldsymbol{W}_{31}^u \\
\boldsymbol{W}_{12}^q = \boldsymbol{W}_{22}^u & & \boldsymbol{W}_{32}^q = \boldsymbol{W}_{32}^u \\
A_{13.5} = \frac{1}{4} & & A_{33.5} = \frac{9}{4}.
\end{array}
$$

All other $A_{\lambda i.5}$ are zero.

Note that no computations with $\lambda = 2$ are needed for case q because $A_{2i.5} = 0$ $(i = 1, 2, 3)$.

The values for the $A_{\lambda i.j}$ given above can be substituted into the formula (4.3) for $\boldsymbol{\Delta}$. This yields a convenient direct computational formula for $\boldsymbol{\Delta}$; no reference to the actual values of the $A_{\lambda i.j}$ is necessary in the applications of R-K 2. Thus (*cf.* also equations (12.2))

$$
\boldsymbol{\Delta} = \begin{pmatrix} h \\ \varphi h \\ \varphi h^2 \left[\frac{1}{32} f(\boldsymbol{U}_0) + \frac{1}{8} f(\boldsymbol{U}_{12}^u) + \frac{27}{64} f(\boldsymbol{U}_{22}^u) + \frac{27}{64} f(\boldsymbol{U}_{32}^u)\right] \\ \varphi h \left[\frac{1}{4} f(\boldsymbol{U}_{12}^p) + \frac{3}{4} f(\boldsymbol{U}_{22}^p)\right] \\ h \left[\frac{1}{4} f(\boldsymbol{U}_{12}^q) + \frac{3}{4} f(\boldsymbol{U}_{32}^q)\right] \end{pmatrix} + \boldsymbol{B}. \tag{10.1}
$$

To obtain this result, use has been made of equations (4.1) and (3.5) as well as the appropriate values of e_λ and g_λ^ω.

§ 11. Determination of U_0 and B_λ^ω

There are a few additional considerations necessary for carrying out the complete numerical process on the whole rectangle $0 \leqq x \leqq a$, $0 \leqq y \leqq b$. The first of these concerns the determination of the initial vector U_0 for subrectangles along the boundaries $x=0$ and $y=0$; the others concern the determination of $\tilde{s}'$, $\tilde{s}''$, $\tilde{t}'$ and $\tilde{t}''$.

For simplicity, it will be assumed here that the mesh is given by $\xi_k = kh$ and $\eta_l = l\varphi h$ where $h = a/m$ and $\varphi h = b/n$ for m, n positive integers. The subrectangle for which $x_0 = \xi_k$, $y_0 = \eta_l$ will be denoted by R_{kl}.

If both $k>0$ and $l>0$, then U_0 is given by the values of u, p and q at $(kh, l\varphi h)$ found by computations on $R_{k-1,l-1}$.

Suppose $k=0$. (The considerations for $l=0$ are similar.) In this case, one can set $u_0 = \tau(l\varphi h)$ and $q_0 = \tau'(l\varphi h)$. To obtain p_0, one can make use of the second of equations (1.5). Indeed, setting

$$\psi(r, p) = f\big(0, r, \tau(r), p, \tau'(r)\big),$$

one has

$$p(0, y) = \sigma'(0) + \int_0^y \psi\big(r, p(0, r)\big)\, dr. \tag{11.1}$$

From this, $p_0 = p(0, l\varphi h)$ can be found by the ordinary Runge-Kutta process. In one form of this process, it is necessary to know the values of τ and τ' not only at the mesh points $j\varphi h$, but also at the intermediate points $(j+\frac{1}{2})\varphi h$ $(j=0, 1, \ldots, n-1)$. Another form of the process uses, instead of the latter, the values of τ and τ' at the points $(j+\frac{1}{3})\varphi h$ and $(j+\frac{2}{3})\varphi h$. This form is perhaps somewhat preferable since the same values are used to find $\tilde{\tau}''$ below.

The quantities $\tilde{s}'$, $\tilde{t}'$, $\tilde{s}''$ and $\tilde{t}''$ are needed in the definition (3.10) of B_λ^ω. They are defined to begin with by equations (3.8). For the subrectangle R_{kl} under consideration, recall that $s'(x_0)$ means $p(\xi_k, \eta_l)$, for example. In the *numerical* procedure the values of $s(x_0+\frac{2}{3}h)$, $t(y_0+\frac{2}{3}\varphi h)$, *etc.*, are not known; however, these can be approximated via Taylor expansions. Equations (3.11), (5.15) and (5.16)—(5.18) show that the resulting approximate values for $\tilde{s}'$, $\tilde{t}'$, $\tilde{s}''$ and $\tilde{t}''$ should be accurate through order h.

Once found, the formulas for computing $\tilde{s}'$, $\tilde{t}'$, $\tilde{s}''$ and $\tilde{t}''$ then become in actuality the *defining* formulas for these quantities; the equations (3.8) must, for the convergence proof of §12, be considered as merely motivation. Among the possible formulas, those given below have been found suitable for use in the convergence proof, and maintain the prescribed accuracy. As a word of caution, it is not clear that for other formulas the convergence proof is valid. For example, if $\tilde{s}''$ is found using u instead of p, then one is led to simultaneous inequalities. Probably $\tilde{s}''$ may be computed via any method of differences in p.

In keeping with the view that the equations below constitute new definitions, equality signs will be used for the most part instead of $\approx$ or an added $\mathcal{O}(h^2)$ to indicate that the quantities of (3.8) are given approximately.

From the original equations (3.8) for $\tilde{s}'$ and $\tilde{t}'$, and from a few terms in the Taylor series and simple forward finite difference substitutions for $s''(x_0)$ and

$t''(y_0)$, one finds on R_{kl} that

$$\tilde{s}' = \tfrac{2}{3}s'(x_0) + \tfrac{1}{3}s'(x_0+h) = \tfrac{2}{3}p(\xi_k, \eta_l) + \tfrac{1}{3}p(\xi_{k+1}, \eta_l),$$
$$\tilde{t}' = \tfrac{2}{3}t'(y_0) + \tfrac{1}{3}t'(y_0+\varphi h) = \tfrac{2}{3}q(\xi_k, \eta_l) + \tfrac{1}{3}q(\xi_k, \eta_{l+1}). \tag{11.2}$$

These equations apply to all subrectangles.

Since in (3.10) the products $h\tilde{s}''$ and $\varphi h\tilde{t}''$ are used, it is these that will be found here. From (3.8)

$$h\tilde{s}'' = \tfrac{3}{2}\left[s'(x_0 + \tfrac{2}{3}h) - s'(x_0)\right]. \tag{11.3}$$

The formula to be given for $h\tilde{s}''$ is to agree with this through order h^2. Expanding $s'(x_0 + \frac{2}{3}h)$ in a Taylor series about x_0 and replacing $s''(x_0)$ and $s'''(x_0)$ by central finite difference formulas, neglecting terms of order h^3, one obtains the result

$$h\tilde{s}'' = \tfrac{1}{6}\left[5s'(x_0+h) - 4s'(x_0) - s'(x_0-h)\right]. \tag{11.4}$$

Similarly,

$$\varphi h\tilde{t}'' = \tfrac{1}{6}\left[5t'(y_0+\varphi h) - 4t'(y_0) - t'(y_0-\varphi h)\right]. \tag{11.5}$$

These equations can be used whenever $k>0$ and $l>0$, respectively.

When $k=0$ and $l=0$, if $\sigma'(\frac{2}{3}h)$ and $\tau'(\frac{2}{3}\varphi h)$ are not known, then one can use the formulas

$$h\tilde{\sigma}'' = \tfrac{1}{24}\left[\sigma'(3h) - 7\sigma'(2h) + 35\sigma'(h) - 29\sigma'(0)\right], \tag{11.6}$$

$$\varphi h\tilde{\tau}'' = \tfrac{1}{24}\left[\tau'(3\varphi h) - 7\tau'(2\varphi h) + 35\tau'(\varphi h) - 29\tau'(0)\right]. \tag{11.7}$$

The equation for $h\tilde{s}''$ when $k=0$ and $l>0$ is again derived using Taylor expansions. Indeed,

$$s'\left(\frac{2}{3}h\right) = s'(0) + \frac{2}{3}h\,s''(0) + \frac{4}{9}\,\frac{h^2}{2}\,s'''(0) + \mathcal{O}(h^3), \tag{11.8}$$

and

$$s'\left(\frac{2}{3}h\right) = s'(h) - \frac{1}{3}h\,s''(h) + \frac{1}{9}\,\frac{h^2}{2}\,s'''(h) + \mathcal{O}(h^3). \tag{11.9}$$

Into (11.8) one may substitute

$$s''(0) = s''(h) - h\,s'''(h) + \mathcal{O}(h^2) \quad \text{and} \quad s'''(0) = s'''(h) + \mathcal{O}(h).$$

Combining the result with (11.9) so as to eliminate the term in $s'''(h)$, one has

$$s'(\tfrac{2}{3}h) = \tfrac{1}{9}\left[s'(0) + 8s'(h) - 2h\,s''(h)\right] + \mathcal{O}(h^3).$$

Here, $s''(h)$ may be replaced by $p_x(h, y_0)$. Then (11.3), through terms of order h^2, yields

$$h\tilde{s}'' = \tfrac{4}{3}\left[s'(h) - s'(0)\right] - \tfrac{1}{3}h\,p_x(h, y_0). \tag{11.10}$$

In this, $s'(h) = p(h, y_0)$ and $s'(0) = p(0, y_0)$ are known from computations on the subrectangle $R_{0,l-1}$.

To determine $p_x(h, y_0)$ through order h^2 two methods are offered. First is the central difference formula

$$p_x(h, y_0) = \frac{p(2h, y_0) - p(0, y_0)}{2h} + \mathcal{O}(h^2)$$

which, substituted into (11.10), yields on $R_{0,l}$

$$h\tilde{s}'' = -\tfrac{1}{6}p(\xi_2, \eta_l) + \tfrac{4}{3}p(\xi_1, \eta_l) - \tfrac{7}{6}p(0, \eta_l). \tag{11.11}$$

This uses the value $p(\xi_2, \eta_l)$ from the computations on $R_{1,l\ 1}$ as well as the results from $R_{0,l-1}$.

Another method for determining $p_x(h, y_0)$ is by an inductive formula:

$$p_x(h, \eta_l) = p_x(h, \eta_{l-1}) + \tfrac{1}{2}\varphi h\left[p_{xy}(h, \eta_l) + p_{xy}(h, \eta_{l-1})\right] + \mathcal{O}(h^3).$$

In this, one may use the original differential equation to replace p_{xy} by $\frac{\partial}{\partial x} f$. Thus

$$p_x(h,\eta_l) = p_x(h, \eta_{l-1}) + \tfrac{1}{2}\varphi\left[F(h, \eta_l) - F(0, \eta_l) + F(h, \eta_{l-1}) - F(0, \eta_{l-1})\right] + \mathcal{O}(h^2), \tag{11.12}$$

in which $F(x, y)$ means $f\big(x, y, u(x, y), p(x, y), q(x, y)\big)$. This does not use any results from computations on $R_{1,l-1}$. The resulting formula for $h\tilde{s}''$ may therefore be somewhat more desirable as regards the overall computational plan for R-K 2.

In an analogous manner, in case $l=0$, one obtains the alternative formulas

$$\varphi h \tilde{t}'' = -\tfrac{1}{6}q(\xi_k, \eta_2) + \tfrac{4}{3}q(\xi_k, \eta_1) - \tfrac{7}{6}q(\xi_k, 0) \tag{11.13}$$

and, if also $k > 0$,

$$\varphi h \tilde{t}'' = \tfrac{4}{3}\left[q(\xi_k, \eta_1) - q(\xi_k, 0)\right] - \tfrac{1}{3}\varphi h\, q_y(\xi_k, \eta_1) \tag{11.14}$$

in which

$$\begin{aligned} q_y(\xi_k, \varphi h) &= q_y(\xi_{k-1}, \varphi h) + \\ &\quad + \frac{1}{2\varphi}\left[F(\xi_k, \varphi h) - F(\xi_k, 0) + F(\xi_{k-1}, \varphi h) - F(\xi_{k-1}, 0)\right]. \end{aligned} \tag{11.15}$$

§ 12. Convergence of the Numerical Approximations Obtained by R-K 2

Here we shall use the notation and terminology defined in [*3*] since the Theorem below is proved by applying I-Theorem 3[3].

The numerical process R-K 2 defines a function from the nodes of a mesh on R into 3-dimensional real space $\mathscr{V}$: to every node point (ξ_k, η_l) is attached the triplet of values $\big(u(\xi_k, \eta_l), p(\xi_k, \eta_l), q(\xi_k, \eta_l)\big)$. This function is clearly a vine as defined in I-§4. The following theorem establishes that $u = u(\xi_k, \eta_l)$ is in fact an approximation to the solution of 2-HP.

It is to be recalled that the regularity assumptions imposed on f in §1 are sufficient to insure the uniqueness of the solution of 2-HP.

Theorem. *Let f, σ and τ satisfy the regularity assumptions of* §1. *If a convergent sequence of meshes be considered in R, and if for each mesh a vine be found by means of* R-K 2 *using the parameter values of* § 10, *then the sequence of vines so determined converges uniformly to the solution of* 2-HP.

Notation. In the foregoing sections, computations on only one subrectangle R_{kl} have been considered. In the following, further subscripts (k, l) will be used to distinguish quantities used on each R_{kl}, for example $\underset{(k,l)}{\boldsymbol{U}_0}$, $\underset{(k,l)}{\boldsymbol{U}^{(\omega)}_{\lambda i}}$, $\underset{(k,l)}{\tilde{s}''}$. Also, $u_{k,l}$ will denote $u(\xi_k, \eta_l)$; $p_{k,l}$ and $q_{k,l}$ are defined similarly.

Proof. In general terms, to prove the assertion, according to I-Theorem 3, two things are to be shown: (1) For each mesh the node function determined

[3] References such as this refer to [*3*].

by R-K 2 can be represented in the *form* of equations I-(4.1) (recall the remark preceding I-Theorem 3):

$$u^\nu_{k+1,l+1} = \sigma^\nu_{k+1} + \tau^\nu_{l+1} - \sigma^\nu_0 + \sum_{\alpha=0}^{k} \sum_{\beta=0}^{l} C^{0\nu}_{\alpha,\beta} \Delta\xi^\nu_\alpha \Delta\eta^\nu_\beta, \tag{12.1 a}$$

$$p^\nu_{k+1,l+1} = \sigma'^\nu_{k+1} + \sum_{\beta=0}^{l} C^{1\nu}_{k+1,\beta} \Delta\eta^\nu_\beta, \tag{12.1 b}$$

$$q^\nu_{k+1,l+1} = \tau'^\nu_{l+1} + \sum_{\alpha=0}^{k} C^{2\nu}_{\alpha,l+1} \Delta\xi^\nu_\alpha. \tag{12.1 c}$$

(2) The quantities $C^{j\nu}_{\alpha,\beta}$ are determined as in I-§4 by functions $\mathscr{C}^{j\nu}(x, y, u, p, q)$ which converge uniformly to $f(x, y, u, p, q)$ on $R\times\mathscr{V}$ and which are constant on each cell of a convergent mesh $\mathscr{M}^{*\nu}$ on $R\times\mathscr{V}$. I-Theorem 3 can then be applied. In the present case, because the solution of 2-HP is unique, the whole sequence of vines (not just a subsequence) must converge to the solution.

The superscript ν will be suppressed throughout the following. Further, for simplicity, the mesh lines in R will be supposed equally spaced: $\xi_k = kh$, $\eta_l = l\varphi h$ where $h=a/m$ and $\varphi h = b/n$ (m, n positive integers). Thus $\Delta\xi_k = h$ and $\Delta\eta_l = \varphi h$. The convergence of the sequence of meshes as $\nu\to\infty$ is assured if $h\to 0$, since for R-K 2 it is assumed that φ is bounded (see §1).

To show (1), one finds from equations (4.4), (10.1) and (3.4) that

$$\begin{aligned} u_{k+1,l+1} &= u_{k+1,l} + u_{k,l+1} - u_{k,l} + \\ &+ \varphi h^2 \big[\tfrac{1}{32} f(\underset{(k,l)}{\boldsymbol{U}_0}) + \tfrac{1}{8} f(\underset{(k,l)}{\boldsymbol{U}^u_{12}}) + \tfrac{27}{64} f(\underset{(k,l)}{\boldsymbol{U}^u_{22}}) + \tfrac{27}{64} f(\underset{(k,l)}{\boldsymbol{U}^u_{32}})\big], \end{aligned} \tag{12.2a}$$

$$p_{k+1,l+1} = p_{k+1,l} + \varphi h \big[\tfrac{1}{4} f(\underset{(k,l)}{\boldsymbol{U}^p_{12}}) + \tfrac{3}{4} f(\underset{(k,l)}{\boldsymbol{U}^p_{22}})\big], \tag{12.2b}$$

$$q_{k+1,l+1} = q_{k,l+1} + h \big[\tfrac{1}{4} f(\underset{(k,l)}{\boldsymbol{U}^q_{12}}) + \tfrac{3}{4} f(\underset{(k,l)}{\boldsymbol{U}^q_{32}})\big]. \tag{12.2c}$$

The three quantities $C^0_{k,l}$, $C^1_{k+1,l}$ and $C^2_{k,l+1}$ are then defined as the coefficients of φh^2, φh and h, respectively, in these three equations. (Note also the subscripts $k+1$ and $l+1$ in the right members of (12.2b) and (12.2c), respectively; they are due to the matrix $\boldsymbol{B}$ in (10.1).)

It is then easy to show by a double induction on k and l (using the natural partial ordering of nodes, *cf.* I-§3) that equation (12.2a) implies that $u_{k+1,l+1}$ is given by an equation of the form (12.1a). Similarly, by induction, p and q are given by equations of the form (12.1b) and (12.1c).

From the form of equations (12.1) and the definitions of the $C^j_{k,l}$, it follows from the boundedness of f that u, p and q are bounded (*cf.* I-§4).

The second thing which is to be shown to prove the Theorem concerns the convergence of the $C^j_{k,l}$ to certain values of f. More specifically, $C^j_{k,l}$ is to converge uniformly to $f(\xi_k, \eta_l, u_{k,l}, p_{k,l}, q_{k,l})$ as $h\to 0$ (*cf.* the statement of Lemma 2 and the definition of $C_{k,l}$ preceeding it in I-§4). From equations (12.2) one obtains the formulas

$$\begin{aligned} C^0_{k,l} &= \tfrac{1}{32} f(\underset{(k,l)}{\boldsymbol{U}_0}) + \tfrac{1}{8} f(\underset{(k,l)}{\boldsymbol{U}^u_{12}}) + \tfrac{27}{64} f(\underset{(k,l)}{\boldsymbol{U}^u_{22}}) + \tfrac{27}{64} f(\underset{(k,l)}{\boldsymbol{U}^u_{32}}), \\ C^1_{k+1,l} &= \tfrac{1}{4} f(\underset{(k,l)}{\boldsymbol{U}^p_{12}}) + \tfrac{3}{4} f(\underset{(k,l)}{\boldsymbol{U}^p_{22}}), \\ C^2_{k,l+1} &= \tfrac{1}{4} f(\underset{(k,l)}{\boldsymbol{U}^q_{12}}) + \tfrac{3}{4} f(\underset{(k,l)}{\boldsymbol{U}^q_{32}}). \end{aligned} \tag{12.3}$$

Note particularly the subscripts $k+1$ and $l+1$ in the last two of these equations. Recall that $\underset{(k,l)}{\boldsymbol{U}_0}$ is the vector notation for $(\xi_k, \eta_l, u_{k,l}, p_{k,l}, q_{k,l})$. Thus, it is to be shown that, as $h \to 0$,

$$
\begin{aligned}
&C^0_{k,l} \to f(\underset{(k,l)}{\boldsymbol{U}_0}),\\
(12.4)\qquad &C^1_{k+1,l} \to f(\underset{(k+1,l)}{\boldsymbol{U}_0}),\\
&C^2_{k,l+1} \to f(\underset{(k,l+1)}{\boldsymbol{U}_0}).
\end{aligned}
$$

By the continuity of f, this will be true if

$$
\begin{aligned}
&\underset{(k,l)}{\boldsymbol{U}^u_{\lambda 2}} \to \underset{(k,l)}{\boldsymbol{U}_0} \qquad (\lambda = 1, 2, 3),\\
(12.5)\qquad &\underset{(k,l)}{\boldsymbol{U}^p_{\lambda 2}} \to \underset{(k+1,l)}{\boldsymbol{U}_0} \qquad (\lambda = 1, 2),\\
&\underset{(k,l)}{\boldsymbol{U}^q_{\lambda 2}} \to \underset{(k,l+1)}{\boldsymbol{U}_0} \qquad (\lambda = 1, 3)
\end{aligned}
$$

uniformly in k and l as $h \to 0$.

With this in mind, we write out explicitly the vectors $\boldsymbol{U}^\omega_{\lambda i}$. From § 10, $\boldsymbol{V}^\omega_\lambda = \boldsymbol{W}^\omega_{\lambda 1} + \boldsymbol{W}^\omega_{\lambda 2} = \boldsymbol{I}$ $(\omega = u, p, q)$. Hence, from (4.1) and (3.9),

$$
(12.6)\quad \underset{(k,l)}{\boldsymbol{U}^u_{\lambda 1}} = \underset{(k,l)}{\boldsymbol{U}_0} + \underset{(k,l)}{\boldsymbol{B}^u_\lambda} + \underset{(k,l)}{\boldsymbol{K}^u_{\lambda 1}} = \begin{pmatrix} \xi_k + g^u_\lambda h \\ \eta_l + e_\lambda g^u_\lambda \varphi h \\ u_{k,l} + g^u_\lambda h \underset{(k,l)}{\tilde{s}'} + e_\lambda g^u_\lambda \varphi h \underset{(k,l)}{\tilde{t}'} + e_\lambda (g^u_\lambda)^2 \varphi h^2 f(\underset{(k,l)}{\boldsymbol{U}_0}) \\ p_{k,l} + g^u_\lambda h \underset{(k,l)}{\tilde{s}''} + e_\lambda g^u_\lambda \varphi h f(\underset{(k,l)}{\boldsymbol{U}_0}) \\ q_{k,l} + e_\lambda g^u_\lambda \varphi h \underset{(k,l)}{\tilde{t}''} + g^u_\lambda h f(\underset{(k,l)}{\boldsymbol{U}_0}) \end{pmatrix}.
$$

The expression for $\boldsymbol{U}^u_{\lambda 2}$ is exactly the same with the sole exception that the argument of f (*viz* $\underset{(k,l)}{\boldsymbol{U}_0}$) is $\underset{(k,l)}{\boldsymbol{U}^u_{\lambda 1}}$.

For $\boldsymbol{U}^p_{\lambda i}$, recall from §10 that $g^p_1 = g^p_2 = 1$. Also $\xi_k + h = \xi_{k+1}$. Hence,

$$
(12.7)\qquad \underset{(k,l)}{\boldsymbol{U}^p_{\lambda 1}} = \underset{(k,l)}{\boldsymbol{U}_0} + \underset{(k,l)}{\boldsymbol{B}^p_\lambda} + \underset{(k,l)}{\boldsymbol{K}^p_{\lambda 1}} = \begin{pmatrix} \xi_{k+1} \\ \eta_l + e_\lambda \varphi h \\ u_{k+1,l} + e_\lambda \varphi h \underset{(k,l)}{\tilde{t}'} + e_\lambda \varphi h^2 f(\underset{(k,l)}{\boldsymbol{U}_0}) \\ p_{k+1,l} + e_\lambda \varphi h f(\underset{(k,l)}{\boldsymbol{U}_0}) \\ q_{k,l} + e_\lambda \varphi h \underset{(k,l)}{\tilde{t}''} + h f(\underset{(k,l)}{\boldsymbol{U}_0}) \end{pmatrix}.
$$

In the last component of this vector, $q_{k,l} + h f(\underset{(k,l)}{\boldsymbol{U}_0})$ can be considered as an approximation to $q_{k+1,l}$ and, in fact, according to (12.2), approaches $q_{k+1,l}$ uniformly since f is bounded. An expression almost identical with that just given for $\boldsymbol{U}^p_{\lambda 1}$ holds also for $\boldsymbol{U}^p_{\lambda 2}$, the only change being that the argument of f becomes $\underset{(k,l)}{\boldsymbol{U}^p_{\lambda 1}}$.

Similar expressions may be found for $\boldsymbol{U}^q_{\lambda i}$.

In order to show (12.5), one sees from (12.6) and (12.7) that it is enough to show that $h\tilde{s}'$, $\varphi h\tilde{t}'$, $h\tilde{s}''$ and $\varphi h\tilde{t}''$ all tend toward zero with h. Indeed, since $|f|<M$, the terms in f cause no trouble. From equations (11.2) defining (for the purposes of the numerical computation) the quantities $\tilde{s}'$ and $\tilde{t}'$, it is clear that $h\tilde{s}'\to 0$ and $\varphi h\tilde{t}'\to 0$ as $h\to 0$ if s' and t' are bounded, that is if p and q are bounded; but this has already been pointed out above.

From equations (11.4) and (11.11) defining $h\tilde{s}''$, if it is shown that $|p_{k+1,l+1}-p_{k,l-1}|$ tends to zero uniformly in k and l as $h\to 0$, then it follows that $h\tilde{s}''\to 0$ uniformly in k and l as $h\to 0$. (If the alternative formula involving (11.12) is used for $h\tilde{s}''$ when $k=0$, difference estimates much like those below yield the same conclusion.) Since the considerations for $\varphi h\tilde{t}''$ are completely analogous, they will not be given here.

From (12.1 b)

$$|p_{k+1,l+1}-p_{k,l+1}|\leqq|\sigma'_{k+1}-\sigma'_k|+\sum_{\beta=0}^{l}|C^1_{k+1,\beta}-C^1_{k,\beta}|\,\Delta\eta_\beta. \tag{12.8}$$

In this, by (12.3) and because the first derivatives of f are bounded, say by L, one has

$$|C^1_{k+1,\beta}-C^1_{k,\beta}|\leqq\tfrac{1}{4}L\,|\underset{(k,\beta)}{\boldsymbol{U}^p_{12}}-\underset{(k-1,\beta)}{\boldsymbol{U}^p_{12}}|+\tfrac{3}{4}L\,|\underset{(k,\beta)}{\boldsymbol{U}^p_{22}}-\underset{(k-1,\beta)}{\boldsymbol{U}^p_{22}}|. \tag{12.9}$$

The vectors $\boldsymbol{U}^p_{\lambda 2}$ are given by (12.7) with the argument of f changed to $\boldsymbol{U}^p_{\lambda 1}$. From this, for $\lambda=1, 2$,

$$\underset{(k,\beta)}{\boldsymbol{U}^p_{\lambda 2}}-\underset{(k-1,\beta)}{\boldsymbol{U}^p_{\lambda 2}} \tag{12.10}$$
$$=\begin{pmatrix} h\\ 0\\ u_{k+1,\beta}-u_{k,\beta}+e_\lambda\varphi h(\underset{(k,\beta)}{\tilde{t}'}-\underset{(k-1,\beta)}{\tilde{t}'})+e_\lambda\varphi h^2[f(\underset{(k,\beta)}{\boldsymbol{U}^p_{\lambda 1}})-f(\underset{(k-1,\beta)}{\boldsymbol{U}^p_{\lambda 1}})]\\ p_{k+1,\beta}-p_{k,\beta}+e_\lambda\varphi h[f(\underset{(k,\beta)}{\boldsymbol{U}^p_{\lambda 1}})-f(\underset{(k-1,\beta)}{\boldsymbol{U}^p_{\lambda 1}})]\\ q_{k,\beta}-q_{k-1,\beta}+e_\lambda\varphi h(\underset{(k,\beta)}{\tilde{t}''}-\underset{(k-1,\beta)}{\tilde{t}''})+h[f(\underset{(k,\beta)}{\boldsymbol{U}^p_{\lambda 1}})-f(\underset{(k-1,\beta)}{\boldsymbol{U}^p_{\lambda 1}})]\end{pmatrix}.$$

From equations (12.1) and the boundedness of f, the differences

$$|u_{k+1,\beta}-u_{k,\beta}| \quad\text{and}\quad |q_{k,\beta}-q_{k-1,\beta}| \tag{12.11}$$

approach zero uniformly in k and β as $h\to 0$. Because f and $\tilde{t}'$ are bounded, the terms in (12.10) involving these approach zero as $h\to 0$.

From equations (11.5) defining $\varphi h\underset{(k,l)}{\tilde{t}''}$ when $k>0$ and $l>0$,

$$\varphi h|\underset{(k,\beta)}{\tilde{t}''}-\underset{(k-1,\beta)}{\tilde{t}''}|\leqq\tfrac{5}{6}|q_{k,\beta+1}-q_{k-1,\beta+1}|+\tfrac{2}{3}|q_{k,\beta}-q_{k-1,\beta}|+\tfrac{1}{6}|q_{k,\beta-1}-q_{k-1,\beta-1}|.$$

The same estimates as those concerning (12.11) thus show that

$$\varphi h|\underset{(k,\beta)}{\tilde{t}''}-\underset{(k-1,\beta)}{\tilde{t}''}|$$

approaches zero uniformly in k and β as $h \to 0$. Similar statements apply when equations (11.13) are used for $\varphi h \tilde{t}''$.

These considerations show that, given ε_1, there is a δ_1 such that if $h < \delta_1$ then in (12.9)

$$|C^1_{k+1,\beta} - C^1_{k,\beta}| \leqq \varepsilon_1 + L\,|p_{k+1,\beta} - p_{k,\beta}|.$$

Thus, in (12.8), given $\varepsilon > 0$ there is a δ such that $h < \delta$ implies

$$|p_{k+1,l+1} - p_{k,l+1}| \leqq \varepsilon + L \sum_{\beta=0}^{l} |p_{k+1,\beta} - p_{k,\beta}|\, \Delta\eta_\beta.$$

Here, the fact that $\sigma'(x)$ is uniformly continuous has been used. By applying Diaz's Lemma (I-Lemma 1) it is then seen that

$$|p_{k+1,l+1} - p_{k,l+1}| \leqq \{\varepsilon + L\,|p_{k+1,0} - p_{k,0}|\, \Delta\eta_0\}\, e^{Lb}.$$

Since $p_{k,0} = \sigma'(\xi_k)$ and $\sigma'(x)$ is uniformly continuous, it follows that $|p_{k+1,l+1} - p_{k,l+1}| \to 0$ uniformly as $h \to 0$. This establishes the desired convergence in (12.5) and hence that in (12.4).

In order to define the functions $\mathscr{C}^j(x, y, u, p, q)$, let $\mathscr{M}$ be the given mesh on R, $\mathscr{M}_{\mathscr{V}}$ a mesh on $\mathscr{V}$, and $\mathscr{M}^*$ the product mesh $\mathscr{M} \times \mathscr{M}_{\mathscr{V}}$ on $R \times \mathscr{V}$ (*cf.* I-§4). On the cell of $\mathscr{M}^*$ containing the point $\underset{(k,l)}{\boldsymbol{U}_0}$ let

$$\mathscr{C}^j(x, y, u, p, q) = C^j_{k,l} \qquad (k = 0, 1, \ldots, m;\ l = 0, 1, \ldots, n).$$

On each remaining cell let a value taken on by f in that cell be chosen. Let the value of $\mathscr{C}^j$ on that cell be the chosen f-value. The convergence expressed by (12.4) then assures that the functions $\mathscr{C}^j$ converge uniformly to f. This completes the proof of the Theorem.

References

[1] Diaz, J. B.: On an analogue of the Euler-Cauchy polygon method for the numerical solution of $u_{xy} = f(x, y, u, u_x, u_y)$. Arch. Rational Mech. Anal. **1**, 357–390 (1958).

[2] Kutta, W.: Beitrag zur näherungsweisen Integration totaler Differentialgleichungen. Z. Math. Phys. **46**, 435–453 (1901).

[3] Moore, R. H.: On approximate solutions of non-linear hyperbolic partial differential equations. Arch. Rational Mech. Anal. **6**, 75–88 (1960).

[4] Prasad, G.: Six Lectures on Mean Value Theorems of Differential Calculus. Univ. of Calcutta, 1931.

[5] Runge, C.: Über die Numerische Auflösung von Differentialgleichungen. Math. Ann. **46**, 167–178 (1895).

Mathematics Research Center-U.S.Army
University of Wisconsin
Madison, Wisconsin

(Received September 29, 1960)

The Invariants of Six Symmetric 3×3 Matrices

A. J. M. Spencer

Communicated by R. S. Rivlin

1. Introduction

In a previous paper [*1*] it was shown that every irreducible invariant, under the orthogonal transformation group, of any number of symmetric 3×3 matrices is necessarily of degree less than or equal to six in the elements of the matrices. In [*1*] and [*2*] the invariants* of five or fewer symmetric 3×3 matrices have been considered, and finite integrity bases have been given for these cases.

This paper will consider the invariants of six symmetric 3×3 matrices, which will be denoted $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$. As shown in [*1*], an integrity basis for these matrices consists of the integrity bases for the six sets of five matrices which can be selected from $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$, together with the invariant

$$\operatorname{tr} \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}\boldsymbol{e}\boldsymbol{f} \tag{1.1}$$

and invariants derived from (1.1) by permuting the matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$. Of the invariants which can be so formed, only a limited number are independent, and the purpose of this paper is to determine an integrity basis for the six matrices which contains the smallest possible number of elements.

Since every irreducible invariant of any number of 3×3 symmetric matrices is of degree six or less in their elements, it follows that each such invariant can involve at most six distinct matrices. Hence an integrity basis for any number of 3×3 matrices consists of the sum of the integrity bases for the matrices taken six at a time in all possible combinations. Thus the results of this paper, with those of the previous papers mentioned above, enable integrity bases for any finite number of symmetric 3×3 matrices to be constructed. It is therefore unnecessary to proceed to study the invariants of seven or more matrices, and this paper completes the analysis of the invariants of 3×3 symmetric matrices.

2. Relations between the invariants

The invariants to be considered are those of the form (1.1) and forms derived from it by permutation of the matrices. Frequent use will be made of the following lemmas, which are immediate consequences of the definitions of matrix multiplication and the trace of a matrix.

* Throughout this paper, invariance is understood to mean invariance with respect to the orthogonal group. It is immaterial whether the full or the proper orthogonal group is considered.

Lemma 1. *The trace of a matrix product is unaltered by a cyclic permutation of the factors of the product.*

Lemma 2. *The trace of a matrix product formed from symmetric matrices is unaltered by reversing the order of the factors in the product.*

There are sixty invariants of six matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$ of the type (1.1) such that no two of the sixty may be equated to each other by applying Lemmas 1 and 2. These sixty may be taken to be

$$\begin{aligned}
&\operatorname{tr}\boldsymbol{abcdef},\ \operatorname{tr}\boldsymbol{abcdfe},\ \operatorname{tr}\boldsymbol{abcedf},\ \operatorname{tr}\boldsymbol{abcefd},\ \operatorname{tr}\boldsymbol{abcfde},\ \operatorname{tr}\boldsymbol{abcfed},\\
&\operatorname{tr}\boldsymbol{abdcef},\ \operatorname{tr}\boldsymbol{abdcfe},\ \operatorname{tr}\boldsymbol{abdecf},\ \operatorname{tr}\boldsymbol{abdefc},\ \operatorname{tr}\boldsymbol{abdfce},\ \operatorname{tr}\boldsymbol{abdfec},\\
&\operatorname{tr}\boldsymbol{abecdf},\ \operatorname{tr}\boldsymbol{abecfd},\ \operatorname{tr}\boldsymbol{abedcf},\ \operatorname{tr}\boldsymbol{abedfc},\ \operatorname{tr}\boldsymbol{abefcd},\ \operatorname{tr}\boldsymbol{abefdc},\\
&\operatorname{tr}\boldsymbol{abfcde},\ \operatorname{tr}\boldsymbol{abfced},\ \operatorname{tr}\boldsymbol{abfdce},\ \operatorname{tr}\boldsymbol{abfdec},\ \operatorname{tr}\boldsymbol{abfecd},\ \operatorname{tr}\boldsymbol{abfedc},\\
&\operatorname{tr}\boldsymbol{acbdef},\ \operatorname{tr}\boldsymbol{acbdfe},\ \operatorname{tr}\boldsymbol{acbedf},\ \operatorname{tr}\boldsymbol{acbefd},\ \operatorname{tr}\boldsymbol{acbfde},\ \operatorname{tr}\boldsymbol{acbfed},\\
&\operatorname{tr}\boldsymbol{acdbef},\ \operatorname{tr}\boldsymbol{acdbfe},\ \operatorname{tr}\boldsymbol{acdebf},\ \operatorname{tr}\boldsymbol{acdfbe},\ \operatorname{tr}\boldsymbol{acebdf},\ \operatorname{tr}\boldsymbol{acebfd},\\
&\operatorname{tr}\boldsymbol{acedbf},\ \operatorname{tr}\boldsymbol{acefbd},\ \operatorname{tr}\boldsymbol{acfbde},\ \operatorname{tr}\boldsymbol{acfbed},\ \operatorname{tr}\boldsymbol{acfdbe},\ \operatorname{tr}\boldsymbol{acfebd},\\
&\operatorname{tr}\boldsymbol{adbcef},\ \operatorname{tr}\boldsymbol{adbcfe},\ \operatorname{tr}\boldsymbol{adbecf},\ \operatorname{tr}\boldsymbol{adbfce},\ \operatorname{tr}\boldsymbol{adcbef},\ \operatorname{tr}\boldsymbol{adcbfe},\\
&\operatorname{tr}\boldsymbol{adcebf},\ \operatorname{tr}\boldsymbol{adcfbe},\ \operatorname{tr}\boldsymbol{adebcf},\ \operatorname{tr}\boldsymbol{adecbf},\ \operatorname{tr}\boldsymbol{adfbce},\ \operatorname{tr}\boldsymbol{adfcbe},\\
&\operatorname{tr}\boldsymbol{aebcdf},\ \operatorname{tr}\boldsymbol{aebdcf},\ \operatorname{tr}\boldsymbol{aecbdf},\ \operatorname{tr}\boldsymbol{aecdbf},\ \operatorname{tr}\boldsymbol{aedbcf},\ \operatorname{tr}\boldsymbol{aedcbf}.
\end{aligned} \tag{2.1}$$

We use the notation

$$\Sigma \boldsymbol{xyz} = \boldsymbol{xyz} + \boldsymbol{xzy} + \boldsymbol{yxz} + \boldsymbol{yzx} + \boldsymbol{zxy} + \boldsymbol{zyx}. \tag{2.2}$$

Also, as in [*1*], [*2*] and [*3*], the notation $I \equiv 0$ will be employed to express the fact that an invariant I is reducible. It was shown in [*3*] that, if $\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are 3×3 matrices, not equal to the unit matrix $\boldsymbol{I}$, then

$$\operatorname{tr} \boldsymbol{w} \Sigma \boldsymbol{xyz} \equiv 0. \tag{2.3}$$

Moreover, all the properties of invariants of 3×3 matrices which were obtained in [*1*], [*2*] and [*3*] may be derived from Lemmas 1 and 2 and relations of the form (2.3).

We will consider all possible relations between the invariants (2.1) which can be obtained by making substitutions in (2.3). First, it is noted that if $\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are any 3×3 matrices (not necessarily symmetric), then

$$\operatorname{tr} \boldsymbol{w} \Sigma \boldsymbol{xyz} = \operatorname{tr} \boldsymbol{x} \Sigma \boldsymbol{yzw} = \operatorname{tr} \boldsymbol{y} \Sigma \boldsymbol{zwx} = \operatorname{tr} \boldsymbol{z} \Sigma \boldsymbol{wxy}. \tag{2.4}$$

These relations are readily verified by introducing relations of the type (2.2) into (2.4) and applying Lemma 1.

In order that (2.3) shall represent a relation between the invariants (2.1), $\boldsymbol{wxyz}$ must be a matrix product which contains each of the factors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$ once and once only, and each of $\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ must contain at least one of the factors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$. There are two possibilities.

(i) One of the products $\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ is a product of three of the factors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$, the remaining three products from $\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ each consisting of a

single factor from $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$. From (2.4) it follows that it may be assumed without loss of generality that the product which contains three factors is the one which precedes the summation sign in (2.3), so that (2.3) takes the form

$$\operatorname{tr}(\boldsymbol{a}\boldsymbol{b}\boldsymbol{c})\,\Sigma\,\boldsymbol{d}\boldsymbol{e}\boldsymbol{f} \equiv 0, \tag{2.5}$$

or a form obtained from this by permuting the matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$.

(ii) Two of the products $\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are products of two factors chosen from $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$, the remaining two products each consisting of a single one of these factors. From (2.4), it may be assumed without loss of generality that the product preceding the summation sign in (2.3) is one of the products which contains two factors, so that (2.3) takes the form

$$\operatorname{tr}(\boldsymbol{a}\boldsymbol{b})\,\Sigma\,(\boldsymbol{c}\boldsymbol{d})\,\boldsymbol{e}\boldsymbol{f} \equiv 0, \tag{2.6}$$

or a form obtained from this by permuting the matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$.

The relations (2.5) and (2.6), and relations derived from them by permuting the matrices, include all possible relations of the type (2.3) between the invariants (2.1). Since all the relations between invariants which can be obtained from results given in [*1*], [*2*] and [*3*] may ultimately be based on (2.3) and Lemmas 1 and 2, it is sufficient to consider relations of the types (2.5) and (2.6), in conjunction with Lemmas 1 and 2, in order to determine how many of the invariants (2.1) are independent. It is, however, convenient to use some further relations which, from the manner in which they are derived, must be expressible as linear combinations of relations of the forms (2.5) and (2.6), although we shall not so express them explicitly. In [*3*] it was shown that, if $\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are defined as in (2.3), then

$$\operatorname{tr}\boldsymbol{x}^2\boldsymbol{y}\boldsymbol{z}^2\boldsymbol{w} \equiv 0, \tag{2.7}$$

and that if $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are also symmetric,

$$\operatorname{tr}\boldsymbol{x}^2\boldsymbol{y}^2\boldsymbol{z}^2 \equiv 0. \tag{2.8}$$

In (2.7), replace $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}$ by $\boldsymbol{a}+\boldsymbol{b}$, $\boldsymbol{e}$, $\boldsymbol{c}+\boldsymbol{d}$, $\boldsymbol{f}$ respectively. Then, on expanding and using the relations (of type (2.7))

$$\operatorname{tr}\boldsymbol{a}^2\boldsymbol{e}\boldsymbol{c}^2\boldsymbol{f} \equiv 0, \quad \operatorname{tr}\boldsymbol{a}^2\boldsymbol{e}\boldsymbol{d}^2\boldsymbol{f} \equiv 0, \quad \operatorname{tr}\boldsymbol{b}^2\boldsymbol{e}\boldsymbol{c}^2\boldsymbol{f} \equiv 0, \quad \operatorname{tr}\boldsymbol{b}^2\boldsymbol{e}\boldsymbol{d}^2\boldsymbol{f} \equiv 0,$$

equation (2.7) becomes

$$\operatorname{tr}(\boldsymbol{a}\boldsymbol{b}+\boldsymbol{b}\boldsymbol{a})\,\boldsymbol{e}\,(\boldsymbol{c}\boldsymbol{d}+\boldsymbol{d}\boldsymbol{c})\,\boldsymbol{f} \equiv 0. \tag{2.9}$$

In a somewhat similar way, we obtain from (2.8) the relation

$$\operatorname{tr}(\boldsymbol{a}\boldsymbol{b}+\boldsymbol{b}\boldsymbol{a})(\boldsymbol{c}\boldsymbol{d}+\boldsymbol{d}\boldsymbol{c})(\boldsymbol{e}\boldsymbol{f}+\boldsymbol{f}\boldsymbol{e}) \equiv 0. \tag{2.10}$$

We will use the notation

$$A(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}) = \operatorname{tr}\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\,\Sigma\,\boldsymbol{d}\boldsymbol{e}\boldsymbol{f}, \tag{2.11}$$

$$B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d}: \boldsymbol{e}, \boldsymbol{f}) = \operatorname{tr}\boldsymbol{a}\boldsymbol{b}\,\Sigma\,(\boldsymbol{c}\boldsymbol{d})\,\boldsymbol{e}\boldsymbol{f}, \tag{2.12}$$

$$C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d}: \boldsymbol{e}, \boldsymbol{f}) = \operatorname{tr}(\boldsymbol{a}\boldsymbol{b}+\boldsymbol{b}\boldsymbol{a})\,\boldsymbol{e}\,(\boldsymbol{c}\boldsymbol{d}+\boldsymbol{d}\boldsymbol{c})\,\boldsymbol{f}, \tag{2.13}$$

$$D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f}) = \operatorname{tr}(\boldsymbol{a}\boldsymbol{b}+\boldsymbol{b}\boldsymbol{a})(\boldsymbol{c}\boldsymbol{d}+\boldsymbol{d}\boldsymbol{c})(\boldsymbol{e}\boldsymbol{f}+\boldsymbol{f}\boldsymbol{e}). \tag{2.14}$$

From the definitions (2.11), (2.12), (2.13) and (2.14), and Lemmas 1 and 2, the following relations follow immediately.

(i)
$$A(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e},\boldsymbol{f}) = A(\boldsymbol{c},\boldsymbol{b},\boldsymbol{a};\boldsymbol{d},\boldsymbol{e},\boldsymbol{f}), \tag{2.15}$$

and $A(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e},\boldsymbol{f})$ is unaltered by permuting $\boldsymbol{d},\boldsymbol{e},\boldsymbol{f}$ in any manner.

(ii)
$$\begin{aligned} B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f}) &= B(\boldsymbol{c},\boldsymbol{d};\boldsymbol{a},\boldsymbol{b}:\boldsymbol{e},\boldsymbol{f}) \\ &= B(\boldsymbol{b},\boldsymbol{a};\boldsymbol{d},\boldsymbol{c}:\boldsymbol{e},\boldsymbol{f}) = B(\boldsymbol{d},\boldsymbol{c};\boldsymbol{b},\boldsymbol{a}:\boldsymbol{e},\boldsymbol{f}), \end{aligned} \tag{2.16}$$

and $B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})$ is unaltered by interchanging $\boldsymbol{e}$ and $\boldsymbol{f}$.

(iii)
$$C(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f}) = C(\boldsymbol{c},\boldsymbol{d};\boldsymbol{a},\boldsymbol{b}:\boldsymbol{e},\boldsymbol{f}), \tag{2.17}$$

and $C(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})$ is unaltered by interchanging $\boldsymbol{a}$ and $\boldsymbol{b}$, or $\boldsymbol{c}$ and $\boldsymbol{d}$, or $\boldsymbol{e}$ and $\boldsymbol{f}$.

(iv)
$$\begin{aligned} D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f}) &= D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{e},\boldsymbol{f};\boldsymbol{c},\boldsymbol{d}) = D(\boldsymbol{c},\boldsymbol{d};\boldsymbol{a},\boldsymbol{b};\boldsymbol{e},\boldsymbol{f}) \\ &= D(\boldsymbol{c},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f};\boldsymbol{a},\boldsymbol{b}) = D(\boldsymbol{e},\boldsymbol{f};\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}) = D(\boldsymbol{e},\boldsymbol{f};\boldsymbol{c},\boldsymbol{d};\boldsymbol{a},\boldsymbol{b}), \end{aligned} \tag{2.18}$$

and $D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f})$ is unaltered by interchanging $\boldsymbol{a}$ and $\boldsymbol{b}$, or $\boldsymbol{c}$ and $\boldsymbol{d}$, or $\boldsymbol{e}$ and $\boldsymbol{f}$.

Taking into account these symmetry properties of $A(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e},\boldsymbol{f})$, $B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})$, $C(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})$ and $D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f})$, there remain for consideration 60 relations of the type $A(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e},\boldsymbol{f}) \equiv 0$, 90 relations of the type $B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f}) \equiv 0$, 45 relations of the type $C(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f}) \equiv 0$, and 15 relations of the type $D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f}) \equiv 0$. These relations are, of course, not all independent.

3. Expressions of the type $A(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e},\boldsymbol{f})$

In this section it will be shown that the expression $A(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e},\boldsymbol{f})$ can be expressed as a linear combination of expressions of the type $C(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})$. We have, from (2.11) and (2.15), and Lemma 2,

$$\begin{aligned} 2A(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e},\boldsymbol{f}) &= \operatorname{tr}(\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}+\boldsymbol{c}\boldsymbol{b}\boldsymbol{a})\,\Sigma\,\boldsymbol{d}\boldsymbol{e}\boldsymbol{f} \\ &= \operatorname{tr}\{\boldsymbol{a}(\boldsymbol{b}\boldsymbol{c}+\boldsymbol{c}\boldsymbol{b}) - \boldsymbol{b}(\boldsymbol{a}\boldsymbol{c}+\boldsymbol{c}\boldsymbol{a}) + \boldsymbol{c}(\boldsymbol{a}\boldsymbol{b}+\boldsymbol{b}\boldsymbol{a})\}\,\Sigma\,\boldsymbol{d}\boldsymbol{e}\boldsymbol{f} \\ &= \operatorname{tr}\{\boldsymbol{a}(\boldsymbol{b}\boldsymbol{c}+\boldsymbol{c}\boldsymbol{b}) - \boldsymbol{b}(\boldsymbol{a}\boldsymbol{c}+\boldsymbol{c}\boldsymbol{a}) + \boldsymbol{c}(\boldsymbol{a}\boldsymbol{b}+\boldsymbol{b}\boldsymbol{a})\}\times \\ &\qquad \times\{\boldsymbol{d}(\boldsymbol{e}\boldsymbol{f}+\boldsymbol{f}\boldsymbol{e}) + \boldsymbol{e}(\boldsymbol{d}\boldsymbol{f}+\boldsymbol{f}\boldsymbol{d}) + \boldsymbol{f}(\boldsymbol{d}\boldsymbol{e}+\boldsymbol{e}\boldsymbol{d})\}, \end{aligned}$$

from (2.2). Hence, by (2.13) and Lemma 1

$$\begin{aligned} 2A(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e},\boldsymbol{f}) &= C(\boldsymbol{b},\boldsymbol{c};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{d}) + C(\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{e}) + C(\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e}:\boldsymbol{a},\boldsymbol{f}) - \\ &\quad - C(\boldsymbol{a},\boldsymbol{c};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{b}\,\boldsymbol{d}) - C(\boldsymbol{a},\boldsymbol{c};\boldsymbol{d},\boldsymbol{f}:\boldsymbol{b},\boldsymbol{e}) - C(\boldsymbol{a},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e}:\boldsymbol{b},\boldsymbol{f}) + \\ &\quad + C(\boldsymbol{a},\boldsymbol{b};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{c},\boldsymbol{d}) + C(\boldsymbol{a},\boldsymbol{b};\boldsymbol{d},\boldsymbol{f}:\boldsymbol{c},\boldsymbol{e}) + C(\boldsymbol{a},\boldsymbol{b};\boldsymbol{d},\boldsymbol{e}:\boldsymbol{c},\boldsymbol{f}). \end{aligned} \tag{3.1}$$

Thus the relations of the form $C(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f}) \equiv 0$ imply the relations $A(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e},\boldsymbol{f}) \equiv 0$, and the latter relations need not be considered further.

4. Expressions of the type $B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})$

We now consider expressions of the form $B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})$. It may be verified by using Lemmas 1 and 2, with (2.2), (2.11), (2.12), (2.13) and (2.14),

that
$$\begin{aligned} &B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f}) + B(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{b}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f}) \\ &\quad = A(\boldsymbol{d}, \boldsymbol{f}, \boldsymbol{a}; \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{e}) + A(\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{a}; \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{f}) - \\ &\quad\quad - A(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a}; \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}) - A(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{a}, \boldsymbol{e}, \boldsymbol{f}) - \\ &\quad\quad - A(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{e}; \boldsymbol{a}, \boldsymbol{d}, \boldsymbol{f}) - A(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{d}, \boldsymbol{e}) + \\ &\quad\quad + 2C(\boldsymbol{b}, \boldsymbol{c}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{d}) + C(\boldsymbol{a}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{b}, \boldsymbol{c}) + C(\boldsymbol{a}, \boldsymbol{d}; \boldsymbol{b}, \boldsymbol{c} : \boldsymbol{e}, \boldsymbol{f}) + \\ &\quad\quad + D(\boldsymbol{a}, \boldsymbol{d}; \boldsymbol{b}, \boldsymbol{c} : \boldsymbol{e}, \boldsymbol{f}). \end{aligned} \tag{4.1}$$

By relations of the form (3.1), the expression on the right-hand side of (4.1) can be expressed as a linear combination of expressions of the types $C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f})$ and $D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f})$.

Also, it may be verified that
$$\begin{aligned} &B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f}) + B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{d}, \boldsymbol{c} : \boldsymbol{e}, \boldsymbol{f}) \\ &\quad = C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f}) + D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f}). \end{aligned} \tag{4.2}$$

From (4.1), (4.2), relations obtained by permuting the matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ in (4.1) and (4.2), and (2.16), it follows that all of the expressions which can be formed from $B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f})$ by permuting the matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ can be expressed as linear combinations of $B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f})$ itself with expressions of the types $C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f})$ and $D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f})$. Thus of the 90 expressions of the form $B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f})$ described in Section 2, 75 may be expressed in terms of the remaining 15 and expressions of the types $C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f})$ and $D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f})$, and need not be considered further. The fifteen expressions which are retained correspond to the fifteen ways in which two matrices can be chosen from the six matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$. Thus $B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f})$ corresponds to the pair $(\boldsymbol{e}, \boldsymbol{f})$, and so on. The fifteen expressions which are retained are taken to be
$$\begin{array}{lll} B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f}), & B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{c}, \boldsymbol{d}), & B(\boldsymbol{b}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{e} : \boldsymbol{a}, \boldsymbol{f}), \\ B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{e} : \boldsymbol{d}, \boldsymbol{f}), & B(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{e} : \boldsymbol{b}, \boldsymbol{f}), & B(\boldsymbol{b}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{e}), \\ B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{f} : \boldsymbol{d}, \boldsymbol{e}), & B(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{f} : \boldsymbol{b}, \boldsymbol{e}), & B(\boldsymbol{b}, \boldsymbol{c}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{d}), \\ B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{d}, \boldsymbol{e} : \boldsymbol{c}, \boldsymbol{f}), & B(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{b}, \boldsymbol{d}), & B(\boldsymbol{b}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{c}), \\ B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{d}, \boldsymbol{f} : \boldsymbol{c}, \boldsymbol{e}), & B(\boldsymbol{a}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{b}, \boldsymbol{c}), & B(\boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{b}). \end{array} \tag{4.3}$$

It may further be verified, by using relations of the types (2.11) and (2.12), that
$$\begin{aligned} &B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f}) + B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{e} : \boldsymbol{d}, \boldsymbol{f}) + B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{f} : \boldsymbol{d}, \boldsymbol{e}) \\ &\quad = A(\boldsymbol{d}, \boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}) + A(\boldsymbol{e}, \boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{f}) + A(\boldsymbol{f}, \boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}). \end{aligned} \tag{4.4}$$

Hence, from (4.4) and relations of the type (3.1),
$$B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f}) + B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{e} : \boldsymbol{d}, \boldsymbol{f}) + B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{f} : \boldsymbol{d}, \boldsymbol{e}) \tag{4.5}$$
can be expressed as a linear combination of expressions of the type $C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f})$.

No essentially new relations between the expressions of the form $B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f})$ can be obtained by permuting $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ in any manner in (4.5). For example, interchanging $\boldsymbol{b}$ and $\boldsymbol{c}$ in (4.5), we find that the expression
$$B(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{b}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f}) + B(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{b}, \boldsymbol{e} : \boldsymbol{d}, \boldsymbol{f}) + B(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{b}, \boldsymbol{f} : \boldsymbol{d}, \boldsymbol{e}) \tag{4.6}$$

can be expressed as a linear combination of expressions of the type $C(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})$. This, however, merely expresses a fact that can be readily deduced from (4.5), expressions of the types (4.1) and (3.1), and the relation (easily verified from (2.11) and (2.14))

$$\begin{aligned}&D(\boldsymbol{a},\boldsymbol{d};\boldsymbol{b},\boldsymbol{c};\boldsymbol{e},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{e};\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{f};\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e})\\&=A(\boldsymbol{b},\boldsymbol{c},\boldsymbol{a};\boldsymbol{d},\boldsymbol{e},\boldsymbol{f})+A(\boldsymbol{b},\boldsymbol{c},\boldsymbol{d};\boldsymbol{a},\boldsymbol{e},\boldsymbol{f})+A(\boldsymbol{b},\boldsymbol{c},\boldsymbol{e};\boldsymbol{a},\boldsymbol{d},\boldsymbol{f})+A(\boldsymbol{b},\boldsymbol{c},\boldsymbol{f};\boldsymbol{a},\boldsymbol{d},\boldsymbol{e}).\end{aligned}\tag{4.7}$$

Again, interchanging $\boldsymbol{a}$ and $\boldsymbol{b}$ in (4.5), it follows that the expression

$$B(\boldsymbol{b},\boldsymbol{a};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})+B(\boldsymbol{b},\boldsymbol{a};\boldsymbol{c},\boldsymbol{e}:\boldsymbol{d},\boldsymbol{f})+B(\boldsymbol{b},\boldsymbol{a};\boldsymbol{c},\boldsymbol{f}:\boldsymbol{d},\boldsymbol{e})\tag{4.8}$$

can be expressed as a linear combination of expressions of the type $C(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})$. This expresses a fact which can be deduced from (4.5), relations of the types (2.16), (3.1) and (4.2), and a relation similar to (4.7). Also, using (2.16), it is evident that (4.4) is unaltered by permuting $\boldsymbol{d}$, $\boldsymbol{e}$ and $\boldsymbol{f}$ in any manner. It follows that of the relations of the form (4.4), it is necessary to consider at most 20 relations, these 20 corresponding to the twenty ways in which three matrices can be chosen from $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$, $\boldsymbol{f}$. Thus (4.4) corresponds to the trio $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$, and so on. All other relations of the form (4.4) can be derived by means of relations already considered from 20 relations selected in this way. The twenty relations retained for further consideration are taken to be

$$\begin{aligned}
&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{e}:\boldsymbol{d},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{f}:\boldsymbol{d},\boldsymbol{e})=\varphi_1,\\
&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{d},\boldsymbol{c}:\boldsymbol{e},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{d},\boldsymbol{e}:\boldsymbol{c},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{d},\boldsymbol{f}:\boldsymbol{c},\boldsymbol{e})=\varphi_2,\\
&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{e},\boldsymbol{c}:\boldsymbol{d},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{e},\boldsymbol{d}:\boldsymbol{c},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{c},\boldsymbol{d})=\varphi_3,\\
&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{f},\boldsymbol{c}:\boldsymbol{d},\boldsymbol{e})+B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{f},\boldsymbol{d}:\boldsymbol{c},\boldsymbol{e})+B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{f},\boldsymbol{e}:\boldsymbol{c},\boldsymbol{d})=\varphi_4,\\
&B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{d},\boldsymbol{b}:\boldsymbol{e},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e}:\boldsymbol{b},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{d},\boldsymbol{f}:\boldsymbol{b},\boldsymbol{e})=\varphi_5,\\
&B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{e},\boldsymbol{b}:\boldsymbol{d},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{e},\boldsymbol{d}:\boldsymbol{b},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{b},\boldsymbol{d})=\varphi_6,\\
&B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{f},\boldsymbol{b}:\boldsymbol{d},\boldsymbol{e})+B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{f},\boldsymbol{d}:\boldsymbol{b},\boldsymbol{e})+B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{f},\boldsymbol{e}:\boldsymbol{b},\boldsymbol{d})=\varphi_7,\\
&B(\boldsymbol{a},\boldsymbol{d};\boldsymbol{e},\boldsymbol{b}:\boldsymbol{c},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{d};\boldsymbol{e},\boldsymbol{c}:\boldsymbol{b},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{b},\boldsymbol{c})=\varphi_8,\\
&B(\boldsymbol{a},\boldsymbol{d};\boldsymbol{f},\boldsymbol{b}:\boldsymbol{c},\boldsymbol{e})+B(\boldsymbol{a},\boldsymbol{d};\boldsymbol{f},\boldsymbol{c}:\boldsymbol{b},\boldsymbol{e})+B(\boldsymbol{a},\boldsymbol{d};\boldsymbol{f},\boldsymbol{e}:\boldsymbol{b},\boldsymbol{c})=\varphi_9,\\
&B(\boldsymbol{a},\boldsymbol{e};\boldsymbol{f},\boldsymbol{b}:\boldsymbol{c},\boldsymbol{d})+B(\boldsymbol{a},\boldsymbol{e};\boldsymbol{f},\boldsymbol{c}:\boldsymbol{b},\boldsymbol{d})+B(\boldsymbol{a},\boldsymbol{e};\boldsymbol{f},\boldsymbol{d}:\boldsymbol{b},\boldsymbol{c})=\varphi_{10},\\
&B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{a}:\boldsymbol{e},\boldsymbol{f})+B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e}:\boldsymbol{a},\boldsymbol{f})+B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{e})=\varphi_{11},\\
&B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{e},\boldsymbol{a}:\boldsymbol{d},\boldsymbol{f})+B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{e},\boldsymbol{d}:\boldsymbol{a},\boldsymbol{f})+B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{d})=\varphi_{12},\\
&B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{f},\boldsymbol{a}:\boldsymbol{d},\boldsymbol{e})+B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{f},\boldsymbol{d}:\boldsymbol{a},\boldsymbol{e})+B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{f},\boldsymbol{e}:\boldsymbol{a},\boldsymbol{d})=\varphi_{13},\\
&B(\boldsymbol{b},\boldsymbol{d};\boldsymbol{e},\boldsymbol{a}:\boldsymbol{c},\boldsymbol{f})+B(\boldsymbol{b},\boldsymbol{d};\boldsymbol{e},\boldsymbol{c}:\boldsymbol{a},\boldsymbol{f})+B(\boldsymbol{b},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{c})=\varphi_{14},\\
&B(\boldsymbol{b},\boldsymbol{d};\boldsymbol{f},\boldsymbol{a}:\boldsymbol{c},\boldsymbol{e})+B(\boldsymbol{b},\boldsymbol{d};\boldsymbol{f},\boldsymbol{c}:\boldsymbol{a},\boldsymbol{e})+B(\boldsymbol{b},\boldsymbol{d};\boldsymbol{f},\boldsymbol{e}:\boldsymbol{a},\boldsymbol{c})=\varphi_{15},\\
&B(\boldsymbol{b},\boldsymbol{e};\boldsymbol{f},\boldsymbol{a}:\boldsymbol{c},\boldsymbol{d})+B(\boldsymbol{b},\boldsymbol{e};\boldsymbol{f},\boldsymbol{c}:\boldsymbol{a},\boldsymbol{d})+B(\boldsymbol{b},\boldsymbol{e};\boldsymbol{f},\boldsymbol{d}:\boldsymbol{a},\boldsymbol{c})=\varphi_{16},\\
&B(\boldsymbol{c},\boldsymbol{d};\boldsymbol{e},\boldsymbol{a}:\boldsymbol{b},\boldsymbol{f})+B(\boldsymbol{c},\boldsymbol{d};\boldsymbol{e},\boldsymbol{b}:\boldsymbol{a},\boldsymbol{f})+B(\boldsymbol{c},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{b})=\varphi_{17},\\
&B(\boldsymbol{c},\boldsymbol{d};\boldsymbol{f},\boldsymbol{a}:\boldsymbol{b},\boldsymbol{e})+B(\boldsymbol{c},\boldsymbol{d};\boldsymbol{f},\boldsymbol{b}:\boldsymbol{a},\boldsymbol{e})+B(\boldsymbol{c},\boldsymbol{d};\boldsymbol{f},\boldsymbol{e}:\boldsymbol{a},\boldsymbol{b})=\varphi_{18},\\
&B(\boldsymbol{c},\boldsymbol{e};\boldsymbol{f},\boldsymbol{a}:\boldsymbol{b},\boldsymbol{d})+B(\boldsymbol{c},\boldsymbol{e};\boldsymbol{f},\boldsymbol{b}:\boldsymbol{a},\boldsymbol{d})+B(\boldsymbol{c},\boldsymbol{e};\boldsymbol{f},\boldsymbol{d}:\boldsymbol{a},\boldsymbol{b})=\varphi_{19},\\
&B(\boldsymbol{d},\boldsymbol{e};\boldsymbol{f},\boldsymbol{a}:\boldsymbol{b},\boldsymbol{c})+B(\boldsymbol{d},\boldsymbol{e};\boldsymbol{f},\boldsymbol{b}:\boldsymbol{a},\boldsymbol{c})+B(\boldsymbol{d},\boldsymbol{e};\boldsymbol{f},\boldsymbol{c}:\boldsymbol{a},\boldsymbol{b})=\varphi_{20},
\end{aligned}\tag{4.9}$$

where the φ_i are linear combinations of expressions of the type $C(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})$.

We next express the relations (4.9) as relations between the fifteen expressions listed in (4.3). Using relations of the types (4.1) and (4.2), equations (4.9) may be written

$$
\begin{aligned}
&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{e}:\boldsymbol{d},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{f}:\boldsymbol{d},\boldsymbol{e})=\psi_1,\\
-&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{d},\boldsymbol{e}:\boldsymbol{c},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{d},\boldsymbol{f}:\boldsymbol{c},\boldsymbol{e})=\psi_2,\\
-&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{e}:\boldsymbol{d},\boldsymbol{f})-B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{d},\boldsymbol{e}:\boldsymbol{c},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{c},\boldsymbol{d})=\psi_3,\\
-&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{f}:\boldsymbol{d},\boldsymbol{e})-B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{d},\boldsymbol{f}:\boldsymbol{c},\boldsymbol{e})-B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{c},\boldsymbol{d})=\psi_4,\\
&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e}:\boldsymbol{b},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{d},\boldsymbol{f}:\boldsymbol{b},\boldsymbol{e})=\psi_5,\\
&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{e}:\boldsymbol{d},\boldsymbol{f})-B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e}:\boldsymbol{b},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{b},\boldsymbol{d})=\psi_6,\\
&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{f}:\boldsymbol{d},\boldsymbol{e})-B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{d},\boldsymbol{f}:\boldsymbol{b},\boldsymbol{e})-B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{b},\boldsymbol{d})=\psi_7,\\
&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{d},\boldsymbol{e}:\boldsymbol{c},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e}:\boldsymbol{b},\boldsymbol{f})+B(\boldsymbol{a},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{b},\boldsymbol{c})=\psi_8,\\
&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{d},\boldsymbol{f}:\boldsymbol{c},\boldsymbol{e})+B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{d},\boldsymbol{f}:\boldsymbol{b},\boldsymbol{e})-B(\boldsymbol{a},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{b},\boldsymbol{c})=\psi_9,\\
&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{c},\boldsymbol{d})+B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{b},\boldsymbol{d})+B(\boldsymbol{a},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{b},\boldsymbol{c})=\psi_{10},\\
-&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})+B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e}:\boldsymbol{a},\boldsymbol{f})+B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{e})=\psi_{11},\\
-&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{e}:\boldsymbol{d},\boldsymbol{f})-B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e}:\boldsymbol{a},\boldsymbol{f})+B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{d})=\psi_{12},\\
-&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{f}:\boldsymbol{d},\boldsymbol{e})-B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{e})-B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{d})=\psi_{13},\\
-&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{d},\boldsymbol{e}:\boldsymbol{c},\boldsymbol{f})+B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e}:\boldsymbol{a},\boldsymbol{f})+B(\boldsymbol{b},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{c})=\psi_{14},\\
-&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{d},\boldsymbol{f}:\boldsymbol{c},\boldsymbol{e})+B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{e})-B(\boldsymbol{b},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{c})=\psi_{15},\\
-&B(\boldsymbol{a},\boldsymbol{b};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{c},\boldsymbol{d})+B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{d})+B(\boldsymbol{b},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{c})=\psi_{16},\\
-&B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e}:\boldsymbol{b},\boldsymbol{f})-B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e}:\boldsymbol{a},\boldsymbol{f})+B(\boldsymbol{c},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{b})=\psi_{17},\\
-&B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{d},\boldsymbol{f}:\boldsymbol{b},\boldsymbol{e})-B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{e})-B(\boldsymbol{c},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{b})=\psi_{18},\\
-&B(\boldsymbol{a},\boldsymbol{c};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{b},\boldsymbol{d})-B(\boldsymbol{b},\boldsymbol{c};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{d})+B(\boldsymbol{c},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{b})=\psi_{19},\\
-&B(\boldsymbol{a},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{b},\boldsymbol{c})-B(\boldsymbol{b},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{c})-B(\boldsymbol{c},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f}:\boldsymbol{a},\boldsymbol{b})=\psi_{20},
\end{aligned}
\tag{4.10}
$$

where the ψ_i are linear combinations of expressions of the types $C(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})$ and $D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})$. The matrix of the coefficients of the 15 expressions (4.3) in the 20 equations (4.10) is

$$
\begin{bmatrix}
1 & -1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
1 & \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
1 & \cdot & \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & 1 & -1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & 1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot & 1 & \cdot & \cdot & -1 & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot & 1 & \cdot & \cdot & -1 & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot & 1 & \cdot & \cdot & -1 & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & 1 & \cdot & \cdot & \cdot & -1\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & 1 & -1
\end{bmatrix}.
\tag{4.11}
$$

It is easily verified that each of the last five rows of this matrix is equal to a linear combination of four out of the first ten rows; for example, the eleventh row is obtained by adding the second row to the seventh row and then sub-

tracting the first and fourth rows. It is also evident from the diagonal submatrix which is formed by the elements common to the first ten rows and last ten columns of the matrix that the first ten rows of the matrix are linearly independent. Hence the matrix has rank ten, and only ten of the equations (4.10), which may be taken to be the last ten, are independent. These equations may be used to express the ten expressions in the first two columns of (4.3) as linear combinations of the five expressions in the third column of (4.3) and expressions of the types $C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f})$ and $D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f})$. Thus each of the original 90 expressions of the form $B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f})$ can be expressed as a linear combination of one or more of the expressions

$$\begin{aligned} &B(\boldsymbol{b}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{e} : \boldsymbol{a}, \boldsymbol{f}), \quad B(\boldsymbol{b}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{e}), \quad B(\boldsymbol{b}, \boldsymbol{c}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{d}), \\ &B(\boldsymbol{b}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{c}), \quad B(\boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{b}), \end{aligned} \tag{4.12}$$

with expressions of the types $C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f})$ and $D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f})$. Hence of the original 90 relations of the type $B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f}) \equiv 0$, it is necessary to consider only the five relations obtained by setting each of the expressions (4.12) equivalent to zero, together with relations of the types $C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f}) \equiv 0$ and $D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f}) \equiv 0$.

5. Expressions of the form $C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f})$

It was shown in Section 2 that it is necessary to give further consideration to 45 expressions of the form $C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f})$, which correspond to the 45 ways in which two pairs of matrices can be chosen from the six matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$, $\boldsymbol{f}$, the ordering of the pairs being immaterial. The expressions to be considered are

$$\begin{array}{lll} C(\boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{b}), & C(\boldsymbol{c}, \boldsymbol{e}; \boldsymbol{d}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{b}), & C(\boldsymbol{c}, \boldsymbol{f}; \boldsymbol{d}, \boldsymbol{e} : \boldsymbol{a}, \boldsymbol{b}), \\ C(\boldsymbol{b}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{c}), & C(\boldsymbol{b}, \boldsymbol{e}; \boldsymbol{d}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{c}), & C(\boldsymbol{b}, \boldsymbol{f}; \boldsymbol{d}, \boldsymbol{e} : \boldsymbol{a}, \boldsymbol{c}), \\ C(\boldsymbol{b}, \boldsymbol{c}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{d}), & C(\boldsymbol{b}, \boldsymbol{e}; \boldsymbol{c}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{d}), & C(\boldsymbol{b}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{e} : \boldsymbol{a}, \boldsymbol{d}), \\ C(\boldsymbol{b}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{e}), & C(\boldsymbol{b}, \boldsymbol{d}; \boldsymbol{c}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{e}), & C(\boldsymbol{b}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{a}, \boldsymbol{e}), \\ C(\boldsymbol{b}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{e} : \boldsymbol{a}, \boldsymbol{f}), & C(\boldsymbol{b}, \boldsymbol{d}; \boldsymbol{c}, \boldsymbol{e} : \boldsymbol{a}, \boldsymbol{f}), & C(\boldsymbol{b}, \boldsymbol{e}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{a}, \boldsymbol{f}), \\ C(\boldsymbol{a}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{b}, \boldsymbol{c}), & C(\boldsymbol{a}, \boldsymbol{e}; \boldsymbol{d}, \boldsymbol{f} : \boldsymbol{b}, \boldsymbol{c}), & C(\boldsymbol{a}, \boldsymbol{f}; \boldsymbol{d}, \boldsymbol{e} : \boldsymbol{b}, \boldsymbol{c}), \\ C(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{b}, \boldsymbol{d}), & C(\boldsymbol{a}, \boldsymbol{e}; \boldsymbol{c}, \boldsymbol{f} : \boldsymbol{b}, \boldsymbol{d}), & C(\boldsymbol{a}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{e} : \boldsymbol{b}, \boldsymbol{d}), \\ C(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{f} : \boldsymbol{b}, \boldsymbol{e}), & C(\boldsymbol{a}, \boldsymbol{d}; \boldsymbol{c}, \boldsymbol{f} : \boldsymbol{b}, \boldsymbol{e}), & C(\boldsymbol{a}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{b}, \boldsymbol{e}), \\ C(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{e} : \boldsymbol{b}, \boldsymbol{f}), & C(\boldsymbol{a}, \boldsymbol{d}; \boldsymbol{c}, \boldsymbol{e} : \boldsymbol{b}, \boldsymbol{f}), & C(\boldsymbol{a}, \boldsymbol{e}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{b}, \boldsymbol{f}), \\ C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{c}, \boldsymbol{d}), & C(\boldsymbol{a}, \boldsymbol{e}; \boldsymbol{b}, \boldsymbol{f} : \boldsymbol{c}, \boldsymbol{d}), & C(\boldsymbol{a}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{e} : \boldsymbol{c}, \boldsymbol{d}), \\ C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{d}, \boldsymbol{f} : \boldsymbol{c}, \boldsymbol{e}), & C(\boldsymbol{a}, \boldsymbol{d}; \boldsymbol{b}, \boldsymbol{f} : \boldsymbol{c}, \boldsymbol{e}), & C(\boldsymbol{a}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{d} : \boldsymbol{c}, \boldsymbol{e}), \\ C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{d}, \boldsymbol{e} : \boldsymbol{c}, \boldsymbol{f}), & C(\boldsymbol{a}, \boldsymbol{d}; \boldsymbol{b}, \boldsymbol{e} : \boldsymbol{c}, \boldsymbol{f}), & C(\boldsymbol{a}, \boldsymbol{e}; \boldsymbol{b}, \boldsymbol{d} : \boldsymbol{c}, \boldsymbol{f}), \\ C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{f} : \boldsymbol{d}, \boldsymbol{e}), & C(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{b}, \boldsymbol{f} : \boldsymbol{d}, \boldsymbol{e}), & C(\boldsymbol{a}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{c} : \boldsymbol{d}, \boldsymbol{e}), \\ C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{e} : \boldsymbol{d}, \boldsymbol{f}), & C(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{b}, \boldsymbol{e} : \boldsymbol{d}, \boldsymbol{f}), & C(\boldsymbol{a}, \boldsymbol{e}; \boldsymbol{b}, \boldsymbol{c} : \boldsymbol{d}, \boldsymbol{f}), \\ C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f}), & C(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{b}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f}), & C(\boldsymbol{a}, \boldsymbol{d}; \boldsymbol{b}, \boldsymbol{c} : \boldsymbol{e}, \boldsymbol{f}). \end{array} \tag{5.1}$$

We denote by S the sum of the sixty invariants listed in (2.1). Then, using (2.13) and Lemmas 1 and 2, it may be verified that

$$\begin{aligned} S = {} & C(\boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{b}) + C(\boldsymbol{c}, \boldsymbol{e}; \boldsymbol{d}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{b}) + C(\boldsymbol{c}, \boldsymbol{f}; \boldsymbol{d}, \boldsymbol{e} : \boldsymbol{a}, \boldsymbol{b}) + \\ & + C(\boldsymbol{b}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{c}) + C(\boldsymbol{b}, \boldsymbol{e}; \boldsymbol{d}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{c}) + C(\boldsymbol{b}, \boldsymbol{f}; \boldsymbol{d}, \boldsymbol{e} : \boldsymbol{a}, \boldsymbol{c}) + \\ & + C(\boldsymbol{b}, \boldsymbol{c}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{d}) + C(\boldsymbol{b}, \boldsymbol{e}; \boldsymbol{c}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{d}) + C(\boldsymbol{b}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{e} : \boldsymbol{a}, \boldsymbol{d}) + \\ & + C(\boldsymbol{b}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{e}) + C(\boldsymbol{b}, \boldsymbol{d}; \boldsymbol{c}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{e}) + C(\boldsymbol{b}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{a}, \boldsymbol{e}) + \\ & + C(\boldsymbol{b}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{e} : \boldsymbol{a}, \boldsymbol{f}) + C(\boldsymbol{b}, \boldsymbol{d}; \boldsymbol{c}, \boldsymbol{e} : \boldsymbol{a}, \boldsymbol{f}) + C(\boldsymbol{b}, \boldsymbol{e}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{a}, \boldsymbol{f}) . \end{aligned} \tag{5.2}$$

Similarly, a further five expressions for S may be obtained by cyclic permutation of the matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$, $\boldsymbol{f}$ in (5.2). It is convenient to use the notation

$$H(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{b}) = C(\boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{b}) + C(\boldsymbol{c}, \boldsymbol{e}; \boldsymbol{d}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{b}) + C(\boldsymbol{c}, \boldsymbol{f}; \boldsymbol{d}, \boldsymbol{e} : \boldsymbol{a}, \boldsymbol{b}) . \tag{5.3}$$

Then (5.2) and the five similar expressions for S may be written

$$\begin{aligned} S &= H(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{b}) + H(\boldsymbol{b}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{c}) + H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{d}) + \\ &\quad + H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{e}) + H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}; \boldsymbol{a}, \boldsymbol{f}) \\ &= H(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{b}) + H(\boldsymbol{a}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{c}) + H(\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{d}) + \\ &\quad + H(\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{e}) + H(\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}; \boldsymbol{b}, \boldsymbol{f}) \\ &= H(\boldsymbol{b}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{c}) + H(\boldsymbol{a}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{c}) + H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{d}) + \\ &\quad + H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{e}) + H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{e}; \boldsymbol{c}, \boldsymbol{f}) \\ &= H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{d}) + H(\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{d}) + H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{d}) + \\ &\quad + H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{f}; \boldsymbol{d}, \boldsymbol{e}) + H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{e}; \boldsymbol{d}, \boldsymbol{f}) \\ &= H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{e}) + H(\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{e}) + H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{e}) + \\ &\quad + H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{f}; \boldsymbol{d}, \boldsymbol{e}) + H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f}) \\ &= H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}; \boldsymbol{a}, \boldsymbol{f}) + H(\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}; \boldsymbol{b}, \boldsymbol{f}) + H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{e}; \boldsymbol{c}, \boldsymbol{f}) + \\ &\quad + H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{e}; \boldsymbol{d}, \boldsymbol{f}) + H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f}) . \end{aligned} \tag{5.4}$$

We now eliminate S from (5.4) and obtain the following five equations between the expressions of the form $H(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{b})$:

$$\begin{aligned} & H(\boldsymbol{b}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{c}) + H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{d}) + H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{e}) + H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}; \boldsymbol{a}, \boldsymbol{f}) - \\ & - H(\boldsymbol{a}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{c}) - H(\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{d}) - H(\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{e}) - H(\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}; \boldsymbol{b}, \boldsymbol{f}) = 0, \\ & H(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{b}) + H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{d}) + H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{e}) + H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}; \boldsymbol{a}, \boldsymbol{f}) - \\ & - H(\boldsymbol{a}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{c}) - H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{d}) - H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{e}) - H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{e}; \boldsymbol{c}, \boldsymbol{f}) = 0, \\ & H(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{b}) + H(\boldsymbol{b}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{c}) + H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{e}) + H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}; \boldsymbol{a}, \boldsymbol{f}) - \\ & - H(\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{d}) - H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{d}) - H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{f}; \boldsymbol{d}, \boldsymbol{e}) - H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{e}; \boldsymbol{d}, \boldsymbol{f}) = 0, \\ & H(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{b}) + H(\boldsymbol{b}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{c}) + H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{d}) + H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}; \boldsymbol{a}, \boldsymbol{f}) - \\ & - H(\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{e}) - H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{e}) - H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{f}; \boldsymbol{d}, \boldsymbol{e}) - H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f}) = 0, \\ & H(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{b}) + H(\boldsymbol{b}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{c}) + H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{d}) + H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{e}) - \\ & - H(\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}; \boldsymbol{b}, \boldsymbol{f}) - H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{e}; \boldsymbol{c}, \boldsymbol{f}) - H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{e}; \boldsymbol{d}, \boldsymbol{f}) - H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f}) = 0. \end{aligned} \tag{5.5}$$

The matrix of the coefficients of the fifteen expressions $H(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{b})$, $H(\boldsymbol{b}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{c}), \ldots, H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f})$ in the five equations (5.5) is

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}. \qquad (5.6)$$

It may be verified that the determinant formed from the first five columns of this matrix is non-zero, and so the matrix has rank five. It follows that equations (5.5) may be used to express five of the expressions of the form $H(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{b})$, which may be taken to be $H(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{b})$, $H(\boldsymbol{b}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{c})$, $H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{d})$, $H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{e})$ and $H(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}; \boldsymbol{a}, \boldsymbol{f})$, in terms of the remaining ten expressions $H(\boldsymbol{a}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{c})$, $H(\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{d}), \ldots, H(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f})$. Since each of the expressions $H(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{b})$ is a sum of three expressions of the type $C(\boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{b})$, and each expression of the form $C(\boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{b})$ occurs in one and only one of the expressions $H(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}; \boldsymbol{a}, \boldsymbol{b})$, equations (5.3) and (5.5) may be used to express five of the expressions $C(\boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{b})$ listed in (5.1) in terms of the remaining forty expressions in (5.1). The five expressions which can be expressed in terms of the other forty can be chosen to be

$$\begin{array}{lll} C(\boldsymbol{c}, \boldsymbol{e}; \boldsymbol{d}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{b}), & C(\boldsymbol{b}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{c}), & C(\boldsymbol{b}, \boldsymbol{e}; \boldsymbol{c}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{d}), \\ C(\boldsymbol{b}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{a}, \boldsymbol{e}), & C(\boldsymbol{b}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{e} : \boldsymbol{a}, \boldsymbol{f}). & \end{array} \qquad (5.7)$$

It follows that the relations which can be obtained by setting the expressions (5.7) equivalent to zero may be derived from the relations obtained by setting equivalent to zero the forty expressions which are included in (5.1) but not in (5.7). Hence, of the relations of the form $C(\boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{b}) \equiv 0$, it is necessary to consider only the forty relations which are obtained by setting equivalent to zero the forty expressions of the type $C(\boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f} : \boldsymbol{a}, \boldsymbol{b})$ which are included in the set (5.1) but not in the set (5.7).

6. Expressions of the type $D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f})$

In this section we consider the fifteen expressions of the form $D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f})$, which correspond to the fifteen ways in which three pairs of matrices can be selected from the six matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$. These fifteen expressions are

$$\begin{array}{lll} D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f}), & D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{e}; \boldsymbol{d}, \boldsymbol{f}), & D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{f}; \boldsymbol{d}, \boldsymbol{e}), \\ D(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{b}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f}), & D(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{b}, \boldsymbol{e}; \boldsymbol{d}, \boldsymbol{f}), & D(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{b}, \boldsymbol{f}; \boldsymbol{d}, \boldsymbol{e}), \\ D(\boldsymbol{a}, \boldsymbol{d}; \boldsymbol{b}, \boldsymbol{c}; \boldsymbol{e}, \boldsymbol{f}), & D(\boldsymbol{a}, \boldsymbol{d}; \boldsymbol{b}, \boldsymbol{e}; \boldsymbol{c}, \boldsymbol{f}), & D(\boldsymbol{a}, \boldsymbol{d}; \boldsymbol{b}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{e}), \\ D(\boldsymbol{a}, \boldsymbol{e}; \boldsymbol{b}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{f}), & D(\boldsymbol{a}, \boldsymbol{e}; \boldsymbol{b}, \boldsymbol{d}; \boldsymbol{c}, \boldsymbol{f}), & D(\boldsymbol{a}, \boldsymbol{e}; \boldsymbol{b}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{d}), \\ D(\boldsymbol{a}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{e}), & D(\boldsymbol{a}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{d}; \boldsymbol{c}, \boldsymbol{e}), & D(\boldsymbol{a}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{e}; \boldsymbol{c}, \boldsymbol{d}). \end{array} \qquad (6.1)$$

It may be verified, by means of relations of types (2.11) and (2.14), that

$$\begin{aligned} &D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{e};\boldsymbol{d},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{f};\boldsymbol{d},\boldsymbol{e})\\ &\quad=A(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e},\boldsymbol{f})+A(\boldsymbol{a},\boldsymbol{b},\boldsymbol{d};\boldsymbol{c},\boldsymbol{e},\boldsymbol{f})+A(\boldsymbol{a},\boldsymbol{b},\boldsymbol{e};\boldsymbol{c},\boldsymbol{d},\boldsymbol{f})+A(\boldsymbol{a},\boldsymbol{b},\boldsymbol{f};\boldsymbol{c},\boldsymbol{d},\boldsymbol{e})\\ &\quad=\chi_1,\end{aligned} \tag{6.2}$$

where, by means of relations of the type (3.1), χ_1 can be expressed as a combination of expressions of the form $C(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})$. Fifteen relations of the type (6.2) may be formulated, corresponding to the fifteen ways in which a pair of matrices can be chosen from the six matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$, $\boldsymbol{f}$. The relations are

$$\begin{aligned}
&D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{e};\boldsymbol{d},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{f};\boldsymbol{d},\boldsymbol{e})=\chi_1,\\
&D(\boldsymbol{a},\boldsymbol{c};\boldsymbol{b},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{c};\boldsymbol{b},\boldsymbol{e};\boldsymbol{d},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{c};\boldsymbol{b},\boldsymbol{f};\boldsymbol{d},\boldsymbol{e})=\chi_2,\\
&D(\boldsymbol{a},\boldsymbol{d};\boldsymbol{b},\boldsymbol{c};\boldsymbol{e},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{d};\boldsymbol{b},\boldsymbol{e};\boldsymbol{c},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{d};\boldsymbol{b},\boldsymbol{f};\boldsymbol{c},\boldsymbol{e})=\chi_3,\\
&D(\boldsymbol{a},\boldsymbol{e};\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{e};\boldsymbol{b},\boldsymbol{d};\boldsymbol{c},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{e};\boldsymbol{b},\boldsymbol{f};\boldsymbol{c},\boldsymbol{d})=\chi_4,\\
&D(\boldsymbol{a},\boldsymbol{f};\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e})+D(\boldsymbol{a},\boldsymbol{f};\boldsymbol{b},\boldsymbol{d};\boldsymbol{c},\boldsymbol{e})+D(\boldsymbol{a},\boldsymbol{f};\boldsymbol{b},\boldsymbol{e};\boldsymbol{c},\boldsymbol{d})=\chi_5,\\
&D(\boldsymbol{a},\boldsymbol{d};\boldsymbol{b},\boldsymbol{c};\boldsymbol{e},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{e};\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{f};\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e})=\chi_6,\\
&D(\boldsymbol{a},\boldsymbol{c};\boldsymbol{b},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{e};\boldsymbol{b},\boldsymbol{d};\boldsymbol{c},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{f};\boldsymbol{b},\boldsymbol{d};\boldsymbol{c},\boldsymbol{e})=\chi_7,\\
&D(\boldsymbol{a},\boldsymbol{c};\boldsymbol{b},\boldsymbol{e};\boldsymbol{d},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{d};\boldsymbol{b},\boldsymbol{e};\boldsymbol{c},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{f};\boldsymbol{b},\boldsymbol{e};\boldsymbol{c},\boldsymbol{d})=\chi_8,\\
&D(\boldsymbol{a},\boldsymbol{c};\boldsymbol{b},\boldsymbol{f};\boldsymbol{d},\boldsymbol{e})+D(\boldsymbol{a},\boldsymbol{d};\boldsymbol{b},\boldsymbol{f};\boldsymbol{c},\boldsymbol{e})+D(\boldsymbol{a},\boldsymbol{e};\boldsymbol{b},\boldsymbol{f};\boldsymbol{c},\boldsymbol{d})=\chi_9,\\
&D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{e};\boldsymbol{b},\boldsymbol{f};\boldsymbol{c},\boldsymbol{d})+D(\boldsymbol{a},\boldsymbol{f};\boldsymbol{b},\boldsymbol{e};\boldsymbol{c},\boldsymbol{d})=\chi_{10},\\
&D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{e};\boldsymbol{d},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{d};\boldsymbol{b},\boldsymbol{f};\boldsymbol{c},\boldsymbol{e})+D(\boldsymbol{a},\boldsymbol{f};\boldsymbol{b},\boldsymbol{d};\boldsymbol{c},\boldsymbol{e})=\chi_{11},\\
&D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{f};\boldsymbol{d},\boldsymbol{e})+D(\boldsymbol{a},\boldsymbol{d};\boldsymbol{b},\boldsymbol{e};\boldsymbol{c},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{e};\boldsymbol{b},\boldsymbol{d};\boldsymbol{c},\boldsymbol{f})=\chi_{12},\\
&D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{f};\boldsymbol{d},\boldsymbol{e})+D(\boldsymbol{a},\boldsymbol{c};\boldsymbol{b},\boldsymbol{f};\boldsymbol{d},\boldsymbol{e})+D(\boldsymbol{a},\boldsymbol{f};\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{e})=\chi_{13},\\
&D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{e};\boldsymbol{d},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{c};\boldsymbol{b},\boldsymbol{e};\boldsymbol{d},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{e};\boldsymbol{b},\boldsymbol{c};\boldsymbol{d},\boldsymbol{f})=\chi_{14},\\
&D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{c};\boldsymbol{b},\boldsymbol{d};\boldsymbol{e},\boldsymbol{f})+D(\boldsymbol{a},\boldsymbol{d};\boldsymbol{b},\boldsymbol{c};\boldsymbol{e},\boldsymbol{f})=\chi_{15},
\end{aligned} \tag{6.3}$$

where each of the χ_i is a sum of expressions of the form $C(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})$.

We now denote the expression on the left-hand side of the first of equations (6.3) by X_1, the expression on the left-hand side of the second of (6.3) by X_2, and so on to X_{15}. Then it may be verified that

$$\begin{aligned}
2X_1&=-X_6-X_7-X_8-X_9+X_{10}+X_{11}+X_{12}+X_{13}+X_{14}+X_{15},\\
2X_2&=-X_6+X_7+X_8+X_9-X_{10}-X_{11}-X_{12}+X_{13}+X_{14}+X_{15},\\
2X_3&=X_6-X_7+X_8+X_9-X_{10}+X_{11}+X_{12}-X_{13}-X_{14}+X_{15},\\
2X_4&=X_6+X_7-X_8+X_9+X_{10}-X_{11}+X_{12}-X_{13}+X_{14}-X_{15},\\
2X_5&=X_6+X_7+X_8-X_9+X_{10}+X_{11}-X_{12}+X_{13}-X_{14}-X_{15}.
\end{aligned} \tag{6.4}$$

Hence, regarding (6.3) as a set of equations for the expressions $D(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c},\boldsymbol{d}:\boldsymbol{e},\boldsymbol{f})$, the first five of equations (6.3) are dependent upon the remaining ten, and need not be considered further. The matrix of the coefficients of the fifteen expressions

listed in (6.1) in the last ten of equations (6.3) is

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{6.5}$$

It can be verified that the determinant of the matrix formed by taking rows 1, 3, 5, 6, 7, 9, 10, 11, 14 and 15 of the matrix (6.5) is non-zero. Hence the matrix (6.5) has rank ten, and it follows that equations (6.3) may be used to express ten of the expressions (6.1) as linear combinations of the remaining five of these expressions and expressions of the form $C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f})$. In particular, (6.3) may be used to express

$$\begin{array}{lll}
D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f}), & D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{f}; \boldsymbol{d}, \boldsymbol{e}), & D(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{b}, \boldsymbol{e}; \boldsymbol{d}, \boldsymbol{f}), \\
D(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{b}, \boldsymbol{f}; \boldsymbol{d}, \boldsymbol{e}), & D(\boldsymbol{a}, \boldsymbol{d}; \boldsymbol{b}, \boldsymbol{c}; \boldsymbol{e}, \boldsymbol{f}), & D(\boldsymbol{a}, \boldsymbol{d}; \boldsymbol{b}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{e}), \\
D(\boldsymbol{a}, \boldsymbol{e}; \boldsymbol{b}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{f}), & D(\boldsymbol{a}, \boldsymbol{e}; \boldsymbol{b}, \boldsymbol{d}; \boldsymbol{c}, \boldsymbol{f}), & D(\boldsymbol{a}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{d}; \boldsymbol{c}, \boldsymbol{e}), \\
 & D(\boldsymbol{a}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{e}; \boldsymbol{c}, \boldsymbol{d}), &
\end{array} \tag{6.6}$$

as linear combinations of

$$\begin{array}{lll}
D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{e}; \boldsymbol{d}, \boldsymbol{f}), & D(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{b}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f}), & D(\boldsymbol{a}, \boldsymbol{d}; \boldsymbol{b}, \boldsymbol{e}; \boldsymbol{c}, \boldsymbol{f}), \\
D(\boldsymbol{a}, \boldsymbol{e}; \boldsymbol{b}, \boldsymbol{f}; \boldsymbol{c}, \boldsymbol{d}), & D(\boldsymbol{a}, \boldsymbol{f}; \boldsymbol{b}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{e}), &
\end{array} \tag{6.7}$$

and expressions of the form $C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f})$. Thus, in place of the fifteen relations of the type $D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f}) \equiv 0$, it is sufficient to consider only the five relations obtained by setting the five expressions (6.7) equivalent to zero, together with relations of the form $C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f}) \equiv 0$.

7. A set of independent invariants

It has been shown in Sections 3—6 that of the relations of the forms

$$A(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}; \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}) \equiv 0, \tag{7.1}$$

$$B(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f}) \equiv 0, \tag{7.2}$$

$$C(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d} : \boldsymbol{e}, \boldsymbol{f}) \equiv 0, \tag{7.3}$$

$$D(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d}; \boldsymbol{e}, \boldsymbol{f}) \equiv 0, \tag{7.4}$$

it is sufficient to retain

(i) Five relations of the form (7.2), which may be taken to be the five relations obtained by setting equivalent to zero the five expressions (4.12);

(ii) Forty relations of the form (7.3), which may be taken to be the forty relations obtained by setting equivalent to zero the forty expressions which appear in (5.1) but not in (5.7);

(iii) Five relations of the type (7.4), which may be taken to be the five relations obtained by setting equivalent to zero five expressions (6.7).

All other relations of the types (7.1) to (7.4) may be regarded as consequences of the relations defined in (i), (ii) and (iii) above. Moreover, (7.1) to (7.4) include all possible relations between the invariants (2.1) which can be derived from (2.3).

It remains to be seen how many of the fifty relations defined above in (i), (ii) and (iii) are independent. In order to do this we form the matrix of the coefficients of the sixty invariants (2.1) in the fifty relations. This matrix is

$$\left[\begin{array}{*{60}{c}}
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & 1 \\
1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . \\
. & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & 1 & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & 1 & . & . & . & 1 & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
1 & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 \\
. & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . \\
. & . & . & 1 & . & 1 & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & 1 & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & 1 & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & . \\
1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . \\
. & . & . & . & . & . & . & 1 & . & . & 1 & . & . & 1 & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & 1 & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . \\
. & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & 1 & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & 1 & 1 & . & . & . & . & . & . \\
1 & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & 1 \\
. & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . \\
1 & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
1 & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & 1 & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & 1 & . & . & . & . & . & . & 1 & 1 & 1 & 1 & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & 1 & 1 & . & . & . & . & . & 1 & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & 1 & . & . & . & . & 1 & 1 & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & 1 & . & 1 \\
1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & 1 & 1
\end{array}\right] . \tag{7.5}$$

In this matrix, the first forty rows are formed from the coefficients of the sixty invariants (2.1) in the forty relations defined in (ii) above; the next five rows are formed from the coefficients of the invariants (2.1) in the five relations defined in (i) above; and the last five rows are formed from the coefficients of the invariants (2.1) in the five relations defined in (iii) above.

It has been verified* that the matrix (7.5) has rank fifty, and that, for example, the determinant of the 50×50 submatrix obtained by omitting columns

$$42,\quad 48,\quad 50,\quad 53,\quad 54,\quad 56,\quad 57,\quad 58,\quad 59,\quad 60, \tag{7.6}$$

from the matrix (7.5) is non-zero. Hence the 50 relations defined in (i), (ii) and (iii) above are all independent, and may be used to express fifty of the invariants (2.1) in terms of the remaining ten invariants (2.1), and only these remaining ten invariants need be retained in the integrity basis. The ten invariants to be retained may be chosen in many ways; for example, they may be taken to be the invariants which correspond to the columns of (7.5) which are listed in (7.6), these invariants being

$$\begin{matrix} \operatorname{tr}\boldsymbol{acfebd}, & \operatorname{tr}\boldsymbol{adcbfe}, & \operatorname{tr}\boldsymbol{adcfbe}, & \operatorname{tr}\boldsymbol{adfbce}, & \operatorname{tr}\boldsymbol{adfcbe}, \\ \operatorname{tr}\boldsymbol{aebdcf}, & \operatorname{tr}\boldsymbol{aecbdf}, & \operatorname{tr}\boldsymbol{aecdbf}, & \operatorname{tr}\boldsymbol{aedbcf}, & \operatorname{tr}\boldsymbol{aedcbf}. \end{matrix} \tag{7.7}$$

An integrity basis for the six 3×3 symmetric matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$ therefore consists of the integrity bases for the matrices taken five at a time, together with the invariants (7.7). Since all possible relations of the type (2.3) between the six matrices have been considered, the number of invariants from (2.1) to be retained in the integrity basis cannot be further reduced by the use of relations of this type, or relations which can be derived from relations of this type. No other polynomial relations between the invariants (2.1) are known to exist.

Acknowledgement. I am very grateful to Professor R. S. Rivlin for introducing the problems described in this paper to me, and for helpful comments and discussion during the preparation of the paper.

* I should like to express my thanks to Mr. J. Nisbet, who has given me a great deal of assistance in making these calculations.

References

[1] Spencer, A. J. M., & R. S. Rivlin: Further results in the theory of matrix polynomials. Arch. Rational Mech. Anal. **4**, 214 (1960).
[2] Spencer, A. J. M., & R. S. Rivlin: Finite integrity bases for five or fewer symmetric 3×3 matrices. Arch. Rational Mech. Anal. **2**, 435 (1959).
[3] Spencer, A. J. M., & R. S. Rivlin: The theory of matrix polynomials and its application to the mechanics of isotropic continua. Arch. Rational Mech. Anal. **2**, 309 (1959).

Department of Theoretical Mechanics
University of Nottingham
England

(Received September 26, 1960)

Bemerkung zum geometrischen Grundgesetz der allgemeinen Kontinuumstheorie der Versetzungen und Eigenspannungen

Ekkehart Kröner

Vorgelegt von J. Meixner

In der vorliegenden Bemerkung, die als Ergänzung zu einer kürzlich erschienenen Arbeit [*1*] aufzufassen ist, soll begründet werden, daß die Einsteinschen Gleichungen $\Gamma^{ij} = B^{ij}$ eine vollständige Formulierung des geometrischen Grundgesetzes der obengenannten Theorie darstellen. Dieses Gesetz beinhaltet die Forderung nach der Kontinuität des verformten Körpers.

In der auf 9 funktionale Freiheitsgrade *beschränkten* Theorie, mit der *reine* Kontinua, d.h. solche ohne Fremd- oder Extramaterie behandelt werden, war das Grundgesetz früher in der linearisierten Form [*2*, *3*]

$$\nabla \times \boldsymbol{\beta} - \boldsymbol{\alpha} = 0 \tag{1}$$

geschrieben worden. Hier ist $\boldsymbol{\beta}$ das Tensorfeld der Distorsion (= Deformation + Drehung) und $\boldsymbol{\alpha}$ das Tensorfeld der Versetzungsdichte. Die auch für endliche Verformungen geltende Formulierung lautet nach Kondo [*4*] sowie Bilby, Bullough und Smith [*5*] (Summationskonvention!)

$$\Gamma_{[ml]k} = A_{\varkappa k} (\partial_m A_l^{\varkappa} - \partial_l A_m^{\varkappa})/2. \tag{2}$$

Hier ist $\Gamma_{[ml]k}$ der in m, l antisymmetrische Teil der affinen Konnexion Γ_{mlk}, die den natürlichen Zustand des Kontinuums in den Koordinaten x^k des Endzustands beschreibt, er repräsentiert die Versetzungsdichte. $A_{\varkappa k}$ ist die Transformation vom natürlichen in den deformierten Zustand, $A_l^{\varkappa}$ die dazu reziproke Transformation. Die letzte Gleichung entsteht aus der Beziehung

$$\Gamma_{mlk} = A_{\varkappa k}\, \partial_m A_l^{\varkappa} \tag{3}$$

durch Antisymmetrisierung bezüglich der Indizes m, l. Wie früher bemerkt ([*1*], S. 289), ist auch Gl. (3) eine Formulierung des Grundgesetzes. Es ist in der Differentialgeometrie wohlbekannt, daß eine Konnexion die Form (3) dann und nur dann hat, wenn der zugehörige Riemann-Christoffelsche Krümmungstensor Γ_{nmlk} verschwindet[1]. Da wir uns, wie früher begründet, auf metrische

[1] Das Verschwinden des Krümmungstensors Γ_{nmlk} in der beschränkten Theorie kommt schon bei Bilby, Bullough und Smith vor.

Kontinua beschränken, ist Γ_{nmlk} in l, k antisymmetrisch und kann somit vollständig durch den zugehörigen Einstein-Tensor Γ^{ij} ersetzt werden. Es folgt, daß die *homogenen* Einsteinschen Gleichungen

$$\Gamma^{ij} = 0 \tag{4}$$

eine weitere Formulierung des geometrischen Grundgesetzes in der beschränkten Theorie darstellen.

Hiernach darf geschlossen werden, daß die *inhomogenen* Einsteinschen Gleichungen

$$\Gamma^{ij} = B^{ij}, \tag{5}$$

in denen B^{ij} wegen der Bianchi-Identität 6 funktionale Freiheitsgrade besitzt, eine vollständige Formulierung des geometrischen Grundgesetzes der allgemeinen Theorie (15 Freiheitsgrade) darstellen, wobei die bei der Verformung hervorgerufenen Kontinuitätsstörungen in der regulären Materie durch die Fremdmaterie (B^{ij}) gerade kompensiert werden.

Wendet man dieselbe Schlußweise auf Gl. (1) an, so kommt man zu einem Gesetz

$$\nabla \times \boldsymbol{\beta} - \boldsymbol{\alpha} = \boldsymbol{\delta}, \tag{6}$$

in dem der die Fremdmaterie repräsentierende Tensor $\boldsymbol{\delta}$ 9 funktionale Freiheitsgrade zu haben scheint. Ziehen wir von Gl. (6) ihre halbe, mit dem Einheitstensor $\boldsymbol{I}$ multiplizierte Spur (Symbol I) ab, so bleibt unter Berücksichtigung des bekannten Zusammenhangs von Versetzungsdichte und Cosserat-Nyeschem Krümmungstensor $\boldsymbol{K}$

$$\nabla \times \boldsymbol{\beta} - \tfrac{1}{2} (\nabla \times \boldsymbol{\beta})_{\mathrm{I}} \boldsymbol{I} + \boldsymbol{K} = \boldsymbol{\delta} - \tfrac{1}{2} \boldsymbol{\delta}_{\mathrm{I}} \boldsymbol{I} \equiv \boldsymbol{\delta}. \tag{7}$$

Mit der früher bewiesenen Zerlegungsformel ([2], S. 29)

$$\boldsymbol{\beta} = \nabla \boldsymbol{s} + \nabla \times \boldsymbol{\iota} \times \nabla + \boldsymbol{I} \times \vec{\Theta}, \tag{8}$$

in der $\boldsymbol{s}$ und $\vec{\Theta}$ je ein Vektorfeld, $\boldsymbol{\iota}$ ein symmetrisches Tensorfeld ist, erhält man

$$\nabla \times (\nabla \times \boldsymbol{\iota} \times \nabla) - \vec{\Theta} \nabla + \boldsymbol{K} = \boldsymbol{\delta}'. \tag{9}$$

Nun entnehmen wir Gl. (9), daß im Fall $\boldsymbol{\delta}' = 0$ und $\boldsymbol{\iota} = 0$ der Krümmungstensor $\boldsymbol{K}$ die Form $\vec{\Theta} \nabla$ hat. Die Realisation erfolgt, wie wir wissen, durch den Angriff einer äußeren Drehmomentendichte. Andere äußere Einflüsse können nun keinesfalls eine Krümmung der gleichen Art, also etwa $\vec{\Theta}' \nabla$ hervorrufen, sie müßten sonst ebenfalls als äußere Drehmomente interpretiert werden (wir denken an unendlich ausgedehnte Medien). Hieraus folgt, daß $\boldsymbol{K}$ die Form

$$\boldsymbol{K} = \vec{\Theta} \nabla + \boldsymbol{\Phi} \times \nabla \tag{10}$$

mit einem noch beliebigen Tensorfeld $\boldsymbol{\Phi}$ hat. Infolgedessen verschwindet die linke Seite von Gl. (9) bei rechtsseitiger Divergenzbildung identisch, und es ist somit $\boldsymbol{\delta}' \cdot \nabla = 0$, d.h. der Tensor $\boldsymbol{\delta}$ verliert 3 seiner funktionalen Freiheitsgrade. Man überzeugt sich leicht, daß andererseits die Gln. (9) bei rechtsseitiger Rotationsbildung in die linearisierten Einsteinschen Gleichungen (5) übergehen ($\boldsymbol{\delta}' \times \nabla = \boldsymbol{B}$).

Damit ist die Gültigkeit unserer Behauptung, daß die Einsteinschen Gleichungen (5) ein vollständiger Ausdruck des geometrischen Grundgesetzes sind, auf anderem Wege bekräftigt worden. Hiermit werden insbesondere auch die in [*1, 6*] zuweilen gebrauchten Bezeichnungen „Grundgleichungen der Eigenspannungsbestimmung" und „Drehmaterie" überflüssig.

Die Beschränkung von $\boldsymbol{\delta}$ auf 6 funktionale Freiheitsgrade bedeutet, daß die Zahl der Freiheitsgrade aller äußeren Einwirkungen nur 12 beträgt, im Gegensatz zu den statischen und geometrischen Zustandsgrößen, die je 15 Freiheitsgrade besitzen. Sind alle äußeren Einwirkungen gleich Null, so gibt es infolge der besonderen Struktur der Einsteinschen Gleichungen trotzdem von Null verschiedene Lösungen, es sind dies die zu den Eigenspannungen führenden Lösungen, als deren Quellen die Versetzungen auftreten. Die hierzu gehörigen Freiheitsgrade sind gerade die 3 überzähligen Freiheitsgrade der geometrischen und statischen Größen. Offensichtlich handelt es sich hier um Lösungen, die metastabile Zustände beschreiben, was in bester Übereinstimmung mit allem ist, was man heute über Versetzungen weiß.

Ich halte es für höchst wahrscheinlich, daß die Einsteinschen Gleichungen überhaupt die einzige vernünftige auf den Endzustand bezogene Formulierung des geometrischen Grundgesetzes der allgemeinen Theorie durch Differentialgleichungen sind, da sie Ableitungen sowohl der Deformationen als auch der Krümmungen enthalten. Darüber hinaus ist die enge Analogie zur allgemeinen Relativitätstheorie sehr ermutigend. Nach den ersten größeren Erfolgen zur Versetzungsdynamik des Kontinuums von Holländer [*7*] darf man hoffen, daß die Einsteinschen Gleichungen (5) auch bei Hinzunahme der 4. Dimension (der Zeit) gültig bleiben, wobei man jetzt die Indizes in (5) von 1 bis 4 laufen lassen sollte.

Literatur

[*1*] Kröner, E.: Allgemeine Kontinuumstheorie der Versetzungen und Eigenspannungen. Arch. Rational Mech. Anal. **4**, 273—334 (1960).

[*2*] Bilby, B. A.: Types of dislocation source, Defects in crystalline solids. Report of 1954 Bristol conference, S. 123—133. London: The Physical Society 1955.

[*3*] Kröner, E.: Kontinuumstheorie der Versetzungen und Eigenspannungen. Erg. angew. Math. **5**, 1—179 (1958).

[*4*] Kondo, K.: Memoirs of the Unifying Study of the Basic Problems in Engineering by Means of Geometry, Vol. I u. II. Tokyo: Gakujutsu Bunken Fukyu-Kai 1955 u. 1958.

[*5*] Bilby, B. A., R. Bullough & E. Smith: Continuous distributions of dislocations: a new application of the methods of non-Riemannian geometry. Proc. Roy. Soc. London, Ser. A **231**, 263—273 (1955).

[*6*] Kröner, E., & A. Seeger: Nicht-lineare Elastizitätstheorie der Versetzungen und Eigenspannungen. Arch. Rational Mech. Anal. **3**, 97—119 (1959).

[*7*] Holländer, E. F.: The basic equations of the dynamics of the continuous distribution of dislocations. Czech. J. Phys. B **10**, 409—418, 479—487, 551—560 (1960).

Institut für theoretische und angewandte Physik
Technische Hochschule Stuttgart
Wiederholdstraße 13

(Eingegangen am 28. Oktober 1960)

A Theory of Dilute Suspensions

G. L. HAND

Communicated by J. L. ERICKSEN

1. Introduction

In this paper the results obtained by JEFFERY [3] for the motion of an ellipsoid suspended in a Newtonian fluid are used to formulate a theory of dilute suspensions. We show that this theory can be considered a special case of ERICKSEN'S theory of anisotropic fluids [1, 2].

2. Anisotropic Fluids

ERICKSEN considers a fluid which is characterized by having a single preferred direction at each point, *e.g.* a suspension of particles of revolution. He lets the preferred direction be described by a vector $\boldsymbol{n}$. ERICKSEN introduces the equations $\varrho \ddot{n}_i = g_i$ where the dot denotes the material derivative. Under the assumptions that

1. the stress tensor t_{ij} and g_i are functions of the type $t_{ij} = t_{ij}(\varrho, n_i, \dot{n}_i, v_{i,j})$, $g_i = g_i(\varrho, n_i, \dot{n}_i, v_{i,j})$, being linear in $\dot{n}_i$ and $v_{i,j}$, where ϱ is the density and v_i is the velocity,

2. the forms of t_{ij} and g_i are preserved under all time-dependent proper orthogonal coordinate transformations, and

3. the structure represented by $\boldsymbol{n}$ is symmetric with respect to reflections in planes parallel and perpendicular to $\boldsymbol{n}$,

he deduces general expressions for t_{ij} and g_i. Additional assumptions lead to simpler forms of t_{ij} and g_i. These are, that the fluid is incompressible, that the particle inertia $\varrho \ddot{n}_i$ is negligible ($g_i = 0$), and that $n_i n_i = 1$. The last assumption restricts the theory to one representing rigid particles. This gives [2, Eqs. (19) and (20)]

$$t_{ij} = -p\,\delta_{ij} + (\lambda_1 + \lambda_2 d_{km} n_k n_m) n_i n_j + 2\lambda_3 d_{ij} + 2\lambda_4 (d_{ik} n_k n_j + d_{jk} n_k n_i), \tag{1}$$

$$\dot{n}_k = \nu (d_{km} n_m - d_{pq} n_p n_q n_k) + w_{km} n_m, \tag{2}$$

where

$$\begin{aligned} w_{ij} &= \tfrac{1}{2}(v_{i,j} - v_{j,i}), \\ d_{ij} &= \tfrac{1}{2}(v_{i,j} + v_{j,i}), \qquad d_{ii} = 0. \end{aligned} \tag{3}$$

The comma denotes differentiation, δ_{ij} is the Kronecker delta, p is an arbitrary isotropic pressure, and ν and the λ's are constants. The ν and λ's might be evaluated by comparison with experiment or with some other theory. We investigate the latter alternative.

3. Jeffery's Calculation

JEFFERY considers a rigid ellipsoidal particle suspended in an incompressible Newtonian fluid. He assumes that inertial and body forces are negligible and that the center of the particle is translated with the mean motion of the fluid. JEFFERY assumes that, except in the immediate neighborhood of the particle, the flow is steady and homogeneous. Laminar flow prevails throughout the flow region, which fills all space exterior to the ellipsoid.

With the simplified Navier-Stokes and continuity equations, JEFFERY calculates the velocity and pressure distributions in the vicinity of the ellipsoid, referred to axes coinciding with the principal axes of the ellipsoid. With these he is able to calculate the resultant couple exerted on the particle. Since inertial forces are assumed negligible, the resultant couple is equated to zero. This provides expressions giving the angular velocity ω_i of the ellipsoid about fixed axes which instantaneously coincide with the principal axes of the ellipsoid. These expressions are [*3*, Eq. (37)]

$$a_{nn}\omega_i - a_{ip}\omega_p = e_{ijk}\,a_{mk}\,v_{m,j}, \tag{4}$$

where e_{ijk} is the alternating tensor and the a_{ij} are the coefficients of the quadratic form describing the ellipsoid. We note that $a_{ij}=0$ for $i \neq j$.

JEFFERY [*3*, Eq. (56)] calculates the stress on the surface of a sphere centered at the suspended particle whose radius R is very large compared to the diameter of the particle. The result is approximate since JEFFERY has neglected terms of order R^{-8} and smaller. This stress, referred to axes coinciding with the principal axes of the ellipsoid, is

$$t_{ij} = -p_0\,\delta_{ij} + 2\mu\,d_{ij} + 10\mu\left\{\frac{5\Phi}{R^5}\,\delta_{ij} + \frac{4x_i\,x_j\,\Phi}{R^7} - \frac{x_i\,\Phi_{,j}}{R^5} - \frac{x_j\,\Phi_{,i}}{R^5}\right\}, \tag{5}$$

where

$$\Phi = A_{pq}\,x_p\,x_q, \tag{6}$$

$$A_{pq} = \left|\begin{matrix} \dfrac{d_{11}}{6\beta_0''} & \dfrac{d_{12}}{2\beta_0'(a^2+b^2)} & \dfrac{d_{13}}{2\beta_0'(a^2+b^2)} \\[2ex] \dfrac{d_{12}}{2\beta_0'(a^2+b^2)} & \dfrac{d_{22}}{4b^2\alpha_0'} + \dfrac{d_{11}(\beta_0''-\alpha_0'')}{12b^2\beta_0''\alpha_0'} & \dfrac{d_{23}}{4b^2\alpha_0'} \\[2ex] \dfrac{d_{13}}{2\beta_0'(a^2+b^2)} & \dfrac{d_{23}}{4b^2\alpha_0'} & \dfrac{d_{33}}{4b^2\alpha_0'} + \dfrac{d_{11}(\beta_0''-\alpha_0'')}{12b^2\beta_0''\alpha_0'} \end{matrix}\right| \tag{7}$$

μ is the coefficient of viscosity, α_0', α_0'', β_0', and β_0'' are given by [*3*, Eq. (10)], and the equation of the ellipsoid of revolution is

$$\frac{x^2}{a^2} + \frac{y^2+z^2}{b^2} = 1. \tag{8}$$

We consider an ellipsoid of revolution because the results may be compared with formulae occurring in the previously mentioned theory of anisotropic fluids. In (5) and (7) the d_{ij} are not the actual rate-of-deformation components of the sphere but the constant values approached as $R\to\infty$. The A_{pq} remain finite as a and b approach zero since each term in A_{pq} is proportional to L^3, L being some characteristic length of the ellipsoid. t_{ij} becomes the Newtonian stress tensor for homogeneous flow of the solvent as the diameter of the suspended particle goes to zero.

4. Theory of Dilute Suspensions

We wish to formulate a theory of dilute suspensions based on the results of JEFFERY'S calculations. If we restrict the theory to

1. incompressible fluids,
2. laminar flow,
3. ellipsoidal suspended particles,
4. a suspension which is very dilute, and
5. negligible particle inertia,

then we should be able to apply JEFFERY'S results.

We must formulate a constitutive equation for the stress. Considering equation (5), we have an approximate relation for the stress at a distance from the particle which is large compared to the particle diameter. This stress is referred to axes coinciding with the principal axes of the ellipsoid. We can average this stress over a volume and get the average stress referred to fixed axes which instantaneously coincide with these principal axes. We interpret this average stress as the macroscopic observable stress of the suspension. We find the space average by using the divergence theorem and noting that $t_{ij,j}=0$.

$$\bar{t}_{ik}=\frac{1}{V}\int_V t_{ik}\,dV=\frac{1}{V}\oint_S x_k\,t_{ij}\,da_j, \tag{9}$$

where V is the volume over which the average is taken and S its surface*. We consider this volume to be a sphere, and, using (5) in (9), we note that the average stress becomes exact as the radius of the sphere becomes infinite. Eq. (9) becomes

$$\begin{aligned}\bar{t}_{ik}=&\frac{1}{V}\oint_S\Big\{-p_0\,\delta_{ij}\,x_k+2\mu\,d_{ij}\,x_k+10\mu\,A_{pq}\Big[\frac{5\,x_p\,x_q\,x_k}{R^5}\,\delta_{ij}+\frac{4\,x_i}{R^7}\,x_j\,x_p\,x_q\,x_k-\\&-\frac{1}{R^5}\,(x_i\,x_k\,x_q\,\delta_{pj}+x_i\,x_k\,x_p\,\delta_{qj}+x_j\,x_k\,x_q\,\delta_{pi}+x_j\,x_k\,x_p\,\delta_{qi})\Big]\Big\}\,da_j\\=&\frac{1}{V}\,(-p_0\,\delta_{ij}+2\mu\,d_{ij})\oint_S x_k\,da_j+\frac{70\mu\,A_{pq}}{V\,R^5}\oint_S x_i\,x_p\,x_q\,da_k-\\&-\frac{10\mu\,A_{pq}}{V\,R^3}\oint_S(x_q\,\delta_{pi}+x_p\,\delta_{qi})\,da_k,\end{aligned} \tag{10}$$

since on the boundary of the sphere p_0, d_{ij}, A_{pq} and R are constants. Using the divergence theorem, we can integrate and obtain

$$\begin{aligned}\oint_S x_k\,da_j&=\int_V x_{k,j}\,dV=\frac{4}{3}\pi R^3\,\delta_{kj},\\\oint_S x_i\,x_p\,x_q\,da_k&=\int_V(x_i\,x_p\,\delta_{qk}+x_i\,x_q\,\delta_{pk}+x_p\,x_q\,\delta_{ik})\,dV\\&=\frac{R}{5}\oint_S(x_i\,x_p\,\delta_{qk}+x_i\,x_q\,\delta_{pk}+x_p\,x_q\,\delta_{ik})\,da\\&=\frac{R^2}{5}\oint_S(x_i\,\delta_{qk}\,da_p+x_i\,\delta_{pk}\,da_q+x_p\,\delta_{ik}\,da_q)\\&=\frac{4\pi R^5}{15}\,(\delta_{ip}\,\delta_{qk}+\delta_{iq}\,\delta_{pk}+\delta_{pq}\,\delta_{ik}).\end{aligned} \tag{11}$$

* We assume that $t_{ij,j}=0$ inside the ellipsoid and that $t_{ij}\,da_j$ is continuous across the surface of the ellipsoid.

Using these results and noting that $A_{ii}=0$, we find that (10) has the form

$$\begin{aligned}\bar{t}_{ij} &= -p_0\,\delta_{ij} + 2\mu\,d_{ij} + \frac{32\pi\mu}{3V}A_{ij},\\ &= -p_0\,\delta_{ij} + 2\mu\,d_{ij} + \frac{8\mu\mathscr{V}}{a\,b^2}A_{ij},\end{aligned} \tag{12}$$

where $\mathscr{V}$ is the ratio of particle volume to fluid volume, namely,

$$\mathscr{V} = \frac{\frac{4}{3}\pi\,a\,b^2}{V} = \frac{a\,b^2}{R^3}. \tag{13}$$

In JEFFERY's stress (5) terms of order R^{-8} and smaller were neglected. In (12) the corresponding terms would be those of order $\mathscr{V}^{\frac{8}{3}}$, which are negligible compared to terms of order $\mathscr{V}$ as long as $\mathscr{V}$ is sufficiently small compared to unity.

Also, we note that we can replace the velocity gradients by their average values, since

$$\bar{v}_{i,j} = \frac{1}{V}\int_V v_{i,j}\,dV = \frac{1}{V}\oint_S x_k\,v_{i,k}\,d\,a_j = v_{i,j}. \tag{14}$$

We have averaged as in (9) above, and $v_{i,j}$ is considered as being constant in regions sufficiently far from the particle. We interpret this average velocity gradient as the macroscopic observable velocity gradient which describes the motion of the suspension.

Equation (12) provides the constitutive equation for the stress for a dilute suspension of particles which satisfy the restrictions mentioned above, at least for cases where the flow of the suspensions is homogeneous. We assume that the same equations are applicable also to inhomogeneous flow.

Now we can show that (12) is just a special case of ERICKSEN's theory of anisotropic fluids. The restrictions presented in §2 give ERICKSEN's stress the simplified form of (1). We choose our coordinate axes in (1) so as to coincide with the principal directions of the ellipsoid and interpret n_i as a unit vector in the direction of the unequal principal axis. Then $n_1=1$, $n_2=n_3=0$, and (1) becomes

$$t_{ij} = -p\,\delta_{ij} + (\lambda_1 + \lambda_2\,d_{11})\,n_i\,n_j + 2\lambda_3\,d_{ij} + 2\lambda_4\,(d_{i1}\,n_j + d_{j1}\,n_i), \tag{15}$$

or, in more detail,

$$\begin{aligned} t_{11} &= -p + \lambda_1 + (\lambda_2 + 2\lambda_3 + 4\lambda_4)\,d_{11},\\ t_{22} &= -p + 2\lambda_3\,d_{22},\\ t_{33} &= -p + 2\lambda_3\,d_{33},\\ t_{12} &= 2(\lambda_3 + \lambda_4)\,d_{21},\\ t_{13} &= 2(\lambda_3 + \lambda_4)\,d_{13},\\ t_{23} &= 2\lambda_3\,d_{23}.\end{aligned} \tag{16}$$

The mean stresses from (12) are

$$\begin{aligned}
\bar{t}_{11} &= -p_0 + \left(2\mu + \frac{4\mu\mathscr{V}}{3ab^2\beta_0''}\right)d_{11},\\
\bar{t}_{22} &= -p_0 + \left(2\mu + \frac{2\mu\mathscr{V}}{ab^4\alpha_0'}\right)d_{22} + \frac{2\mu\mathscr{V}(\beta_0''-\alpha_0'')}{3ab^4\beta_0''\alpha_0'}\,d_{11},\\
\bar{t}_{33} &= -p_0 + \left(2\mu + \frac{2\mu\mathscr{V}}{ab^4\alpha_0'}\right)d_{33} + \frac{2\mu\mathscr{V}(\beta_0''-\alpha_0'')}{3ab^4\alpha_0''\beta_0'}\,d_{11},\\
\bar{t}_{12} &= \left(2\mu + \frac{4\mu\mathscr{V}}{ab^2\beta_0'(a^2+b^2)}\right)d_{12},\\
\bar{t}_{13} &= \left(2\mu + \frac{4\mu\mathscr{V}}{ab^2\beta_0'(a^2+b^2)}\right)d_{13},\\
\bar{t}_{23} &= \left(2\mu + \frac{2\mu\mathscr{V}}{ab^2\alpha_0'}\right)d_{23}.
\end{aligned} \tag{17}$$

For comparison we assume that the d_{ij}'s occurring in (16) and (17) both describe the motion of the suspension and that the t_{ij} of (16) and $\bar{t}_{ij}$ of (17) both represent the stress in the suspension. If we compare $\bar{t}_{23}$ and t_{23} and let $\lambda_3=\mu(1+\sigma)$, we have

$$\sigma = \frac{\mathscr{V}}{ab^4\alpha_0'}. \tag{18}$$

Comparing $\bar{t}_{12}$ and t_{12} or $\bar{t}_{13}$ and t_{13}, we have

$$\lambda_4 = \frac{\mu\mathscr{V}}{ab^2}\left(\frac{2}{\beta_0'(a^2+b^2)} - \frac{1}{b^2\alpha_0'}\right). \tag{19}$$

Comparing $\bar{t}_{22}$ and t_{22} or $\bar{t}_{33}$ and t_{33}, we have

$$p = p_0 + \frac{2\mu\mathscr{V}(\alpha_0''-\beta_0'')\,d_{11}}{3ab^4\alpha_0'\beta_0''} = p_0 + \frac{2\mu\mathscr{V}(\alpha_0''-\beta_0'')}{3ab^4\alpha_0'\beta_0''}\,d_{ij}n_in_j. \tag{20}$$

In the theory of anisotropic fluids, p is an arbitrary pressure. Finally comparing $\bar{t}_{11}$ and t_{11}, we have

$$\begin{aligned}
\lambda_2 &= \frac{2\mu\mathscr{V}}{ab^2}\left(\frac{\alpha_0''}{b^2\alpha_0'\beta_0''} + \frac{1}{b^2\alpha_0'} - \frac{4}{\beta_0'(a^2+b^2)}\right),\\
\lambda_1 &= 0.
\end{aligned} \tag{21}$$

Thus either (1) with the constants as just evaluated or (12) may be used as the constitutive equation for the stress in this dilute suspension.

We also wish to know the trajectory of a particle. Since the particle inertia is assumed negligible, the center of the particle will be translated with the mean motion of the surrounding fluid. The rotation of the particle is given by (2) or (4). These give the angular velocities about axes which instantaneously coincide with the principal axes of the ellipsoid. To see that (2) and (4) are equivalent, we note that the angular velocity occurring in (4) is defined so that $\dot{n}_i=e_{ijk}\omega_j n_k$. We can write ω_i as the resultant of a vector perpendicular to n_i and one parallel to it, the part perpendicular to n_i being given by

$$\omega_i^{\perp} = e_{ijk}n_j\dot{n}_k. \tag{22}$$

Then (2) becomes

$$\omega_i^{\perp} = e_{ijk}(\nu\,d_{kp} + w_{kp})\,n_p n_j. \tag{23}$$

The theory of anisotropic fluids does not account for the component of ω_i parallel to n_i. Comparing (23) with (4), we find

$$\nu = \frac{a^2 - b^2}{a^2 + b^2}, \tag{24}$$

where a is the length of the unequal half-axis and b is the length of the equal half-axes for the ellipsoid of revolution. Hence ν is negative for disk-like particles and positive for rod-like particles. Equation (4) or (2) with the constant evaluated in (24) provides the trajectory of a suspended ellipsoid of revolution.

We now have the necessary equations for the laminar motion of a dilute suspension of ellipsoids of revolution in an incompressible fluid. The standard form of the equations of conservation of mass and momentum give

$$d_{ii} = 0, \qquad \varrho \dot{v}_i = t_{ij,j} + f_i, \tag{25}$$

where f_i is the body force. The stress is found from (12) or (1) with the constants evaluated above, and the angular velocity of a particle is provided by (2) or (4).

This work was supported by Grant NSF-G 11316 from the U.S. National Science Foundation.

References

[1] Ericksen, J. L.: Anisotropic fluids. Arch. Rational Mech. Anal. **4**, 231–237 (1959).
[2] Ericksen, J. L.: Transversely isotropic fluids (to appear in Kolloid-Z.).
[3] Jeffery, G. B.: The motion of ellipsoidal particles immersed in a viscous fluid. Proc. Roy. Soc. A **102**, 161–179 (1922).

Department of Mechanics
The Johns Hopkins University
Baltimore, Maryland

(Received October 17, 1960)

EDITORIAL BOARD

CONTENTS

Erdélyi, A., General Asymptotic Expansions of Laplace Integrals . . 1

Karim, R. I. I. A., Über den Resonanzbegriff bei Systemen von n linearen gewöhnlichen Differentialgleichungen erster Ordnung 21

Karim, R. I. I. A., Über den Resonanzfall bei Systemen von n nichtlinearen gewöhnlichen Differentialgleichungen erster Ordnung . . . 29

Moore, R. H., A Runge-Kutta Procedure for the Goursat Problem in Hyperbolic Partial Differential Equations 37

Spencer, A. J. M., The Invariants of Six Symmetric 3×3 Matrices . . 64

Kröner, E., Bemerkung zum geometrischen Grundgesetz der allgemeinen Kontinuumstheorie der Versetzungen und Eigenspannungen . 78

Hand, G. L., A Theory of Dilute Suspensions 81

Druck der Universitätsdruckerei H. Stürtz AG., Würzburg
Printed in Germany